

# Tighter Regret Bounds for Influence Maximization and Other Combinatorial Semi-Bandits with Probabilistically Triggered Arms

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## Abstract

We study combinatorial multi-armed bandit with probabilistically triggered arms (CMAB-T) and semi-bandit feedback. We resolve a serious issue in the prior CMAB-T studies where the regret bounds contain a possibly exponentially large factor of  $1/p^*$ , where  $p^*$  is the minimum positive probability that an arm is triggered by any action. We address this issue by introducing triggering probability moderated (TPM) bounded smoothness conditions into the general CMAB-T framework, and show that many applications such as influence maximization bandit and combinatorial cascading bandit satisfy such TPM conditions. As a result, we completely remove the factor of  $1/p^*$  from the regret bounds, achieving significantly better regret bounds for influence maximization and cascading bandits than before. Finally, we provide lower bound results showing that the factor  $1/p^*$  is unavoidable for general CMAB-T problems, suggesting that TPM conditions are crucial in removing this factor.

## 1. Introduction

Multi-armed bandit (MAB) is a classical online learning framework modeled as a game between a player and the environment with  $m$  arms. In each round, the player selects one arm and the environment generates a reward of the arm, either stochastically from a distribution unknown to the player or in an adversarial way. The player observes the reward, which is used as the feedback to the player's algorithm (or policy) to select arms in future rounds. The goal of the player is to cumulate as much reward as possible over time. MAB models the classical tradeoff between exploration and exploitation in online learning and sequential decision making: the player needs to explore new arms that may give better rewards, but also need to exploit the

best arm observed so far to ensure that the expected cumulative reward is close to the best possible one. The standard performance measure of the player's algorithm is the (*expected*) *regret*, which is the difference in expected cumulative reward between always playing the best arm in expectation and playing according to the player's algorithm. MAB and its variants have been extensively studied. One influential algorithm for stochastic MAB is the Upper Confidence Bound (UCB) algorithm (Auer et al., 2002a), which has regret  $O(\sum_i \frac{1}{\Delta_i} \log T)$  distribution-dependent regret and  $O(\sqrt{mT \log T})$  distribution-independent regret, where  $\Delta_i$  is the gap in expected reward between the best arm and arm  $i$ , and  $T$  is the number of rounds.

In recent years, stochastic combinatorial multi-armed bandit (CMAB) receives many attention (e.g. (Gai et al., 2012; Chen et al., 2016b;a; Gopalan et al., 2014; Kveton et al., 2014; 2015b;a;c; Combes et al., 2015)), because it has wide applications in wireless networking, online advertising and recommendation, viral marketing in social networks, etc., which demand the integration of combinatorial optimization with sequential online learning. In the typical setting of CMAB, the player selects an abstract action to play in each round, which would trigger the play of a set of arms, and the outcomes of these triggered arms are observed as the feedback (called semi-bandit feedback). Besides the exploration and exploitation tradeoff, CMAB also needs to deal with the exponential explosion of the possible actions such that even exploring every action once is infeasible.

One class of the above CMAB problems involves probabilistically triggered arms (Chen et al., 2016b; Kveton et al., 2015a;c), in which actions may trigger arms probabilistically. We denote it as CMAB-T in this paper. Chen et al. (2016b) provide such a general model and the regret bounds for a UCB-style algorithm CUCB, and apply it to the influence maximization bandit, which models stochastic influence diffusion in social networks and sequentially selecting seed node sets to maximize the cumulative influence spread over time. Kveton et al. (2015a;c) study cascading bandits, in which the action is to select a sequence of arms, which will be probabilistically triggered following the sequence order based on the outcome of the already triggered arms. They essentially

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use the same algorithm as CUCB and provide regret upper bounds for disjunctive and conjunctive cascading bandits.

However, both studies left an important issue unaddressed. In (Chen et al., 2016b), the regret bounds contain an undesirable factor of  $1/p^*$ , where  $p^*$  is the minimum positive probability that any arm can be triggered by any action. For influence maximization bandit,  $1/p^*$  could be exponentially large to the problem instance. For combinatorial cascading bandit (Kveton et al., 2015c), a similar factor  $1/f^*$  exists in the regret bounds, the value of which depends on the means of the arms and could also be exponentially small, especially for disjunctive cascading bandits.

In this paper, we decisively address the above issue by completely removing the above undesirable terms from the influence maximization bandit and the combinatorial cascading bandit. Instead of fixing the specific influence maximization or cascading bandit, we adapt the general CMAB framework of (Chen et al., 2016b) in a systematic way to reach the solution. The key intuition to our solution is as follows. First, for a generic CMAB-T problem,  $1/p^*$  may be unavoidable, since we may have to trigger some particular arms to see their feedback in order to find the best action, but on average it takes  $1/p^*$  to trigger such arms once. In fact, we provide a lower bound result in Section 5 to show that  $1/p^*$  is unavoidable for general CMAB-T problems. However, for influence maximization and cascading bandits, there is a desirable property that if an arm is hard to trigger, its contribution to the expected reward is also small, and thus we may not need to observe it as often as others. We turn this key observation into a triggering probability moderated (TPM) bounded smoothness condition on the expected reward function, adapting from the original bounded smoothness condition in (Chen et al., 2016b). We show that conjunctive and disjunctive cascading bandits satisfy this condition, while for the influence maximization bandit, we need a weaker version called *restricted* TPM (RTPM). While the existing CUCB algorithm works with the TPM condition, we need to extend it to CUCB-IS (standing for intentional sampling) to work with the RTPM condition.

We analyze the CUCB/CUCB-IS algorithm, and provide distribution-dependent and distribution-independent regret bounds. There is no triggering probability related factors in the regret bound, which means the factor  $1/p^*$  and  $1/f^*$  are removed from influence maximization bandit and cascading bandit, respectively. For other problem instance dependent factors in the dominant  $\sum_i \frac{1}{\Delta_i} \log T$  or  $\sqrt{T \log T}$  term, ours are the same as in (Kveton et al., 2015c) for cascading bandit. As for the influence maximization bandit, we further improve the result in (Chen et al., 2016b) by a factor of  $|E|$  for the distribution-dependent regret and a fac-

tor of  $\sqrt{|E|}$  for the distribution-independent regret ( $|E|$  is the number of edges in the social graph), due to our use of the tighter 1-norm bounded smoothness conditions while Chen et al. (2016b) use infinity-norm conditions.

Besides removing the exponential factor in the regret bounds, our analysis is also tighter in that the leading constants in regret bounds are much smaller than the ones in (Kveton et al., 2015c) for combinatorial cascading bandits, and the ones in (Kveton et al., 2015b) for linear bandits without probabilistically triggered arms. We also extend the CMAB-T model in (Chen et al., 2016b) to accommodate infinite action space and explicit probabilistic triggering distributions.

Due to the space constraint, complete proofs are moved to the supplementary material.

### 1.1. Related Work

Multi-armed bandit problem is originally formed by Robbins (1952), and has been extensively studied in the literature (cf. Berry & Fristedt, 1985; Sutton & Barto, 1998; Bubeck & Cesa-Bianchi, 2012). Our study belongs to the stochastic bandit research, while there is another line of research on adversarial bandits (Auer et al., 2002b), for which we refer to a survey like (Bubeck & Cesa-Bianchi, 2012) for further information. For stochastic MABs, besides the celebrated UCB approach (Auer et al., 2002a), there is also the Bayesian style Thompson sample approach (Thompson, 1933), but its theoretical regret bound analysis comes much later (Agrawal & Goyal, 2012). Lai & Robbins (1985) show the asymptotic lower bound of  $\Omega(\sum_i \frac{1}{\Delta_i} T)$  for the distribution-dependent regret, while Auer et al. (1995) show the lower bound of  $\Omega(\sqrt{mT})$  for the distribution-independent regret.

As already mentioned in the introduction, stochastic CMAB has received many attention in recent years. Among the studies, we improve (a) the general framework with probabilistically triggered arms of (Chen et al., 2016b), (b) the influence maximization bandit results in (Chen et al., 2016b) and (Wen et al., 2016), (c) the combinatorial cascading bandit results in (Kveton et al., 2015c), and (d) the linear bandit results in (Kveton et al., 2015b). We defer the technical comparison with these studies to Section 4.1. Other CMAB studies do not deal with probabilistically triggered arms. Among them, (Gai et al., 2012) is the first study on linear stochastic bandit, but its regret bound has since been improved by Chen et al. (2016b); Kveton et al. (2015b). Combes et al. (2015) improve the regret bound of (Kveton et al., 2015b) for linear bandits in a special case where arms are mutually independent. Chen et al. (2016a) provide a UCB-style algorithm and regret analysis for general reward functions for which only

estimating the mean outcomes of arms is not enough. [Gopalan et al. \(2014\)](#) study Thompson sampling for complex actions, which can be applied to CMAB, but their regret bound has a large exponential constant term.

Influence maximization is first formulated as a discrete optimization problem by [Kempe et al. \(2003\)](#), and has been extensively studied since (cf. [\(Chen et al., 2013\)](#)). Variants of influence maximization bandit have also been studied by [\(Lei et al., 2015; Vaswani et al., 2015\)](#). [Lei et al. \(2015\)](#) use a different objective of maximizing the expected size of the union of the influenced nodes over time, while [Vaswani et al. \(2015\)](#) discuss node level feedback rather than the edge level feedback. However, as for regret minimization, their results are mostly empirical and there is no theoretical regret bounds for us to compare with.

## 2. General Framework

Combinatorial multi-armed bandit with probabilistically triggered arms is a class of stochastic online learning problems originally proposed by [Chen et al. \(2016b\)](#). In this section we present the general framework of [\(Chen et al., 2016b\)](#) with a slight adaptation, and denote it as CMAB-T. We illustrate that the influence maximization bandit [\(Chen et al., 2016b\)](#) and combinatorial cascading bandits [\(Kveton et al., 2015a;c\)](#) are example instances of CMAB-T.

### 2.1. CMAB-T Framework

CMAB-T is described as a learning game between a learning agent (or player) and the environment. The environment consists of  $m$  random variables  $X_1, \dots, X_m$  called *base arms* (or *arms*) following a joint distribution  $D$  over  $[0, 1]^m$ . Distribution  $D$  is picked by the environment from a class of distributions  $\mathcal{D}$  before the game starts. Crucially the player knows  $\mathcal{D}$  but not the actual distribution  $D$ .

The learning process proceeds in discrete rounds. In round  $t \geq 1$ , the player selects an action  $S_t$  from an action space  $\mathcal{S}$  based on the feedback history from the previous rounds, and the environment draws from the joint distribution  $D$  an independent sample  $X^{(t)} = (X_1^{(t)}, \dots, X_m^{(t)})$ . When action  $S_t$  is played on the environment outcome  $X^{(t)}$ , a random subset of arms  $\tau_t \subseteq [m]$  are triggered, and the outcomes of  $X_i^{(t)}$  for all  $i \in \tau_t$  are observed as the feedback to the player. The player also obtains a nonnegative reward  $R(S_t, X^{(t)}, \tau_t)$  fully determined by  $S_t, X^{(t)}$ , and  $\tau_t$ . A learning algorithm aims at properly selecting actions  $S_t$ 's over time based on the past feedback to cumulate as much reward as possible. Different from [\(Chen et al., 2016b\)](#), we allow the action space  $\mathcal{S}$  to be infinite.

We now describe the triggered set  $\tau_t$  in more detail. In gen-

eral,  $\tau_t$  may have additional randomness beyond the randomness of  $X^{(t)}$ . Let  $D^{\text{trig}}(S, X)$  denote a distribution of the triggered subset of  $[m]$  for a given action  $S$  and an environment outcome  $X$ . We assume that  $\tau_t$  is drawn independently from  $D^{\text{trig}}(S_t, X^{(t)})$ . We refer  $D^{\text{trig}}$  as the *probabilistic triggering function*.

To summarize, a *CMAB-T problem instance* is a tuple  $([m], \mathcal{S}, \mathcal{D}, D^{\text{trig}}, R)$ , where  $[m] = \{1, 2, \dots, m\}$  is the set of arms,  $\mathcal{S}$  is the set of feasible actions,  $\mathcal{D}$  is the class of distributions of the environment over  $[0, 1]^m$ ,  $D^{\text{trig}}$  is a probabilistic triggering function, and  $R$  is the reward function determined by action  $S \in \mathcal{S}$ , environment outcome  $X \in [0, 1]^m$  and triggered set  $\tau \subseteq [m]$ . All elements in the above tuple are known to the player, and hence establishing the problem input to the player. In contrast, the *environment instance* is the actual distribution  $D \in \mathcal{D}$  picked by the environment, and is unknown to the player. The problem instance and the environment instance together form the *(learning) game instance*, in which the learning process would unfold. In this paper, we fix the environment instance  $D$ , unless we need to refer to more than one environment instances.

For each arm  $i$ , let  $\mu_i = \mathbb{E}_{X \sim D}[X_i]$ . Let vector  $\mu = (\mu_1, \dots, \mu_m)$  denote the expectation vector of base arms. Same as in [\(Chen et al., 2016b\)](#), we assume that the expected reward  $\mathbb{E}[R(S, X, \tau)]$ , where the expectation is taken over  $X \sim D$  and  $\tau \sim D^{\text{trig}}(S, X)$ , is a function of action  $S$  and the expectation vector  $\mu$  of the arms. Henceforth, we denote  $r_\mu(S) \triangleq \mathbb{E}[R(S, X, \tau)]$ . For convenience, we also use the notation  $r_S(\mu)$  for  $r_\mu(S)$ .

The performance of a learning algorithm  $A$  is measured by its *(expected) regret*, which is the difference in expected cumulative reward between always playing the best action and playing actions selected by algorithm  $A$ . Formally, let  $\text{opt}_\mu = \sup_{S \in \mathcal{S}} r_\mu(S)$ , where  $\mu = \mathbb{E}_{X \sim D}[X]$ , and we assume that  $\text{opt}_\mu$  is finite. Same as in [\(Chen et al., 2016b\)](#), we assume that the learning algorithm has access to an offline  $(\alpha, \beta)$ -approximation oracle  $\mathcal{O}$ , which takes  $\mu = (\mu_1, \dots, \mu_m)$  as input and outputs an action  $S^\mathcal{O}$  such that  $\Pr\{r_\mu(S^\mathcal{O}) \geq \alpha \cdot \text{opt}_\mu\} \geq \beta$ , where  $\alpha$  is the *approximation ratio* and  $\beta$  is the success probability. Under the  $(\alpha, \beta)$ -approximation oracle, the benchmark cumulative reward should be the  $\alpha\beta$  fraction of the optimal reward, and thus we use the following  $(\alpha, \beta)$ -approximation regret:

**Definition 1** ( $(\alpha, \beta)$ -approximation Regret). *The  $T$ -round  $(\alpha, \beta)$ -approximation regret of a learning algorithm  $A$  (using an  $(\alpha, \beta)$ -approximation oracle) for a CMAB-T game instance  $([m], \mathcal{S}, \mathcal{D}, D^{\text{trig}}, R, D)$  with  $\mu = \mathbb{E}_{X \sim D}[X]$  is*

$$\text{Reg}_{\mu, \alpha, \beta}^A(T) = T \cdot \alpha \cdot \beta \cdot \text{opt}_\mu - \mathbb{E} \left[ \sum_{i=1}^T r_\mu(S_i^A) \right],$$

where  $S_t^A$  is the action  $A$  selects in round  $t$ , and the expectation is taken over the randomness of the environment outcomes  $X^{(1)}, \dots, X^{(T)}$ , the triggered sets  $\tau_1, \dots, \tau_T$ , as well as the possible randomness of algorithm  $A$  itself.

The above framework essentially follows (Chen et al., 2016b), but we decouple actions from subsets of base arms, allow action space to be infinite, and explicitly model triggered set distribution, which makes the framework more powerful in modeling certain applications (see supplementary material for more detailed discussions).

## 2.2. Influence Maximization and Cascading Bandits

Both the influence maximization bandits (Chen et al., 2016b) and combinatorial cascading bandits (Kveton et al., 2015a;c) are examples of CMAB-T.

In social influence maximization (Kempe et al., 2003), we are given a weighted directed graph  $G = (V, E, p)$ , where  $V$  and  $E$  are sets of vertices and edges respectively, and each edge  $(u, v)$  is associated with a probability  $p(u, v)$ . Starting from a seed set  $S \subseteq V$ , influence propagates in  $G$  as follows: nodes in  $S$  are activated at time 0, and at time  $t \geq 1$ , a node  $u$  activated in step  $t - 1$  has one chance to activate its inactive out-neighbor  $v$  with an independent probability  $p(u, v)$ . The *influence spread* of seed set  $S$ ,  $\sigma(S)$ , is the expected number of activated nodes after the propagation ends. The offline problem of *influence maximization* is to find at most  $k$  seed nodes in  $G$  such that the influence spread is maximized. Kempe et al. (2003) provide a greedy algorithm with approximation ratio  $1 - 1/e - \varepsilon$  and success probability  $1 - 1/|V|$ , for any  $\varepsilon > 0$ .

For the online influence maximization bandit (Chen et al., 2016b), the edge probabilities  $p(u, v)$ 's are unknown and need to be learned over time through repeated influence maximization tasks. Putting it into the CMAB-T framework, the set of edges  $E$  is the set of arms  $[m]$ , and their outcome distribution  $D$  is the joint distribution of  $m$  independent Bernoulli distributions with means  $p(u, v)$  for all  $(u, v) \in E$ . Any seed set  $S \subseteq V$  with at most  $k$  nodes is an action. Given action  $S_t$  and outcome  $X^{(t)} = \{X_e^{(t)}\}_{e \in E}$  in the  $t$ -th influence maximization task, all edges from  $S_t$  that are reached by the propagation are triggered and their outcomes are observed. That is, the triggered arm set  $\tau_t$  is the set of edges  $(u, v)$  such that  $u$  can be reached from  $S_t$  by passing through only edges  $e \in E$  with  $X_e^{(t)} = 1$ . In this case, the distribution  $D^{\text{trig}}(S_t, X^{(t)})$  degenerates to a deterministic triggered set. The reward  $R(S_t, X^{(t)}, \tau_t)$  equals to the number of nodes in  $V$  that is reached from  $S$  through only edges  $e \in E$  with  $X_e = 1$ , and the expected reward is exactly the influence spread  $\sigma(S_t)$ . The offline oracle is a  $(1 - 1/e - \varepsilon, 1/|V|)$ -approximation greedy algorithm. We remark that the general triggered set distri-

bution  $D^{\text{trig}}(S_t, X^{(t)})$  (together with infinite action space) can be used to model extended versions of influence maximization, such as randomly selected seed sets in general marketing actions (Kempe et al., 2003) and continuous influence maximization (Yang et al., 2016) (see supplementary material).

Now let us consider combinatorial cascading bandits (Kveton et al., 2015a;c). In this case, we have  $m$  independent Bernoulli random variables  $X_1, \dots, X_m$  as base arms. An action is to select an ordered sequence from a subset of these arms, such as any sequence of exactly  $k$  arms. Playing this action means letting a user go through the sequence in the given order, revealing the outcomes of the arms one by one until certain stopping condition is satisfied. The feedback is the outcomes of revealed arms and the reward is a function form of these arms. In particular, in the disjunctive form the user stops when the first 1 is revealed as the arm outcome or she reaches the end of the sequence with all 0 outcomes; in the first case the reward is 1 and in the second case the reward is 0. In the conjunctive form, the user stops when the first 0 is revealed (and receives reward 0) or she reaches the end with all 1 outcomes (and receives reward 1). It is straightforward to see that cascading bandits fit into the CMAB-T framework:  $m$  variables are base arms, ordered sequences are actions, and the triggered set is the prefix set of arms until the stopping condition holds.

## 3. Triggering-Probability Moderated Conditions

Chen et al. (2016b) use two conditions to guarantee the theoretical regret bounds. The first one is monotonicity, which we also use in this paper, and is restated below.

**Condition 1 (Monotonicity).** *We say that a CMAB-T problem instance satisfies monotonicity, if the expected reward of playing any action  $S \in \mathcal{S}$  is monotonically non-decreasing with respect to the expectation vector, i.e., for any two distributions  $D, D' \in \mathcal{D}$  with expectation vectors  $\mu = (\mu_1, \dots, \mu_m)$  and  $\mu' = (\mu'_1, \dots, \mu'_m)$ , we have  $r_S(\mu) \leq r_S(\mu')$  for all  $S \in \mathcal{S}$  if  $\mu_i \leq \mu'_i$  for all  $i \in [m]$ .*

The second condition is bounded smoothness. One key contribution of our paper is to properly strengthen the original bounded smoothness condition in (Chen et al., 2016b) so that we can both get rid of the undesired  $1/p^*$  term in the regret bound and guarantee that many CMAB problems such as influence maximization and cascading bandits still satisfy the conditions. Our important change is to use triggering probabilities to moderate the condition, and thus we call such conditions *triggering probability moderated (TPM)* conditions. In this section, we further use 1-norm based conditions instead of the infinity-norm based condi-



tion in (Chen et al., 2016b), since they lead to better regret bounds for the influence maximization and cascading bandits. In the supplementary material, we also provide conditions and associated regret bound results in  $\infty$ -norm, since in general the two sets of results do not imply each other.

For simplicity, we assume that the probability of triggering base arm  $i$  with action  $S$ ,  $\Pr_{X \sim D, \tau \sim D^{\text{trig}}(S, X)}(i \in \tau)$ , is determined by the expectation vector  $\mu$  (instead of the full distribution  $D$ ), together with  $S$  (and  $D^{\text{trig}}$ ). This certainly holds when  $X_1, \dots, X_n$  are mutually independent Bernoulli random variables, since in this case  $\mu$  fully determines  $D$ . Thus, for now, we denote  $p_i^{\mu, S} \triangleq \Pr_{X \sim D, \tau \sim D^{\text{trig}}(S, X)}(i \in \tau)$ . When the context is clear, we may ignore  $\mu$ ,  $S$  in the super script. Let  $p_i^{\mu, *} = \inf_{S \in \mathcal{S}, p_i^{\mu, S} > 0} p_i^{\mu, S}$ , and if  $\{S \in \mathcal{S} \mid p_i^{\mu, S} > 0\} = \emptyset$ , we define  $p_i^{\mu, *} = 1$ . Let  $p^{\mu, *} = \min_{i \in [m]} p_i^{\mu, *}$ .

**Condition 2 (1-Norm TPM Bounded Smoothness).** We say that a CMAB-T problem instance satisfies 1-norm TPM bounded smoothness, if there exists  $B \in \mathbb{R}^+$  (referred as the bounded smoothness constant) such that, for any two distributions  $D, D' \in \mathcal{D}$  with expectation vectors  $\mu$  and  $\mu'$ , and any action  $S$ , we have  $|r_S(\mu) - r_S(\mu')| \leq B \sum_{i \in [m]} p_i^{\mu, S} |\mu_i - \mu'_i|$ .

Note that the corresponding non-TPM version of the above condition would remove  $p_i^{\mu, S}$  in the above condition, which is a generalization of the linear condition used in linear bandits (Kveton et al., 2015b). Thus, the TPM version is clearly stronger than the non-TPM version (when the bounded smoothness constants are the same). The intuition of incorporating the triggering probability  $p_i^{\mu, S}$  to moderate the 1-norm condition is that, when an arm  $i$  is unlikely triggered by action  $S$  (small  $p_i^{\mu, S}$ ), the importance of arm  $i$  also diminishes in that a large change in  $\mu_i$  only causes a small change in the expected reward  $r_S(\mu)$ . This property sounds natural in many applications, and it is important for bandit learning — although an arm  $i$  may be difficult to observe when playing  $S$ , it also means that it is not important to the expected reward of  $S$  and thus does not need to be learned as accurately as others more easily triggered by  $S$ . The following lemma shows that the cascading bandit satisfies this property.

**Lemma 1.** For both disjunctive and conjunctive cascading bandit problem instances, 1-norm TPM bounded smoothness (Condition 2) holds with bounded smoothness constant  $B = 1$ .

Intuitively, the influence maximization bandit also has the above property that a less likely triggered arm (edge in this case) is less important for the expected reward, but technically, we need the following restricted TPM (or RPTM) version for the influence maximization bandit.

**Condition 3 (1-Norm RTPM Bounded Smoothness).**

We say that a CMAB-T problem instance satisfies 1-norm RTPM bounded smoothness, if there exists bounded smoothness constant  $B \in \mathbb{R}^+$  such that, for any two distributions  $D, D' \in \mathcal{D}$  with expectation vectors  $\mu$  and  $\mu'$ , and any action  $S$ , if  $\max_{i \in [m]} |\mu_i - \mu'_i| \leq \frac{1}{2m}$ , we have  $|r_S(\mu) - r_S(\mu')| \leq B \sum_{i \in [m]} p_i^{\mu, S} |\mu_i - \mu'_i|$ .

**Lemma 2.** Influence maximization bandit problem instances satisfy 1-norm RTPM bounded smoothness (Condition 3) with bounded smoothness constant  $B = 2|V|$ .

We remark that the proof of the above lemma is nontrivial. In fact, we introduce two general conditions local to each arm  $i$  that together imply 1-norm RPTM bounded smoothness, and then show that the influence maximization bandit satisfies these two conditions. These two conditions are of independent interest, but due to space constraint, we move them to the supplementary material.

## 4. CUCB/CUCB-IS Algorithm

We present our CUCB/CUCB-IS algorithm in Algorithm 1. It has two working modes differentiated by a flag named `IntentionalSampling`. When `IntentionalSampling` is set to `false`, it is the same as the CUCB algorithm in (Chen et al., 2016b). This algorithm fits CMAB-T problem instances that satisfy non-restricted version of bounded smoothness conditions (Condition 2). When `IntentionalSampling` is set to `true`, we call the algorithm CUCB-IS (IS stands for intentional sampling), which is designed to fit CMAB-T problem instances that satisfy *restricted* TPM bounded smoothness (Condition 3). The difference between two working modes is that CUCB-IS sometimes intentionally samples arms to satisfy the additional restriction  $|\mu_i - \mu'_i| \leq \frac{1}{2m}$ . To achieve this, for CUCB-IS we additionally assume that there is an easy way to choose an action  $S$  for each arm  $i$ , such that  $p_i^S = 1$ . For influence maximization bandit, each arm is an edge  $(u, v)$ , and thus the above requirement is achieved simply by selecting a seed set containing  $u$ .

### Regret bound for CUCB/CUCB-IS algorithm

**Definition 2 (Gap).** Fix a distribution  $D$  and its expectation vector  $\mu$ . For each action  $S$ , we define the gap  $\Delta_S = \max(0, \alpha \cdot \text{opt} - r_S(\mu))$ . For each arm  $i$ , we define

$$\Delta_{\min}^i = \inf_{S \in \mathcal{S}: p_i^{\mu, S} > 0, \Delta_S > 0} \Delta_S,$$

$$\Delta_{\max}^i = \sup_{S \in \mathcal{S}: p_i^{\mu, S} > 0, \Delta_S > 0} \Delta_S.$$

As a convention, if there is no action  $S$  such that  $p_i^{\mu, S} > 0$  and  $\Delta_S > 0$ , we define  $\Delta_{\min}^i = +\infty$ ,  $\Delta_{\max}^i = 0$ . We define  $\Delta_{\min} = \min_{i \in [m]} \Delta_{\min}^i$ , and  $\Delta_{\max} = \max_{i \in [m]} \Delta_{\max}^i$ .

**Algorithm 1** CUCB/CUCB-IS with computation oracle.

**Input:**  $m$ , Oracle, IntentionalSampling

- 1: For each arm  $i$ ,  $T_i \leftarrow 0$  {maintain the total number of times arm  $i$  is played so far}
- 2: For each arm  $i$ ,  $\hat{\mu}_i \leftarrow 1$  {maintain the empirical mean value of  $X_i$ }
- 3:  $t \leftarrow 0$
- 4: **loop**
- 5:    $t \leftarrow t + 1$
- 6:   For each arm  $i \in [m]$ ,  $\rho_i \leftarrow \sqrt{\frac{3 \ln t}{2T_i}}$  {the radius of the confidence interval,  $\rho_i = +\infty$  if  $T_i = 0$ }
- 7:   **if** IntentionalSampling **and**  $\exists i \in [m]$ ,  $\rho_i \geq \frac{1}{4m}$  **then**
- 8:      $S \leftarrow$  An action  $S$  that  $p_i^S = 1$
- 9:   **else**
- 10:     For each arm  $i \in [m]$ ,  $\bar{\mu}_i = \min \{\hat{\mu}_i + \rho_i, 1\}$  {the upper confidence bound}
- 11:      $S \leftarrow$  Oracle( $\bar{\mu}_1, \dots, \bar{\mu}_m$ )
- 12:   **end if**
- 13:   Play action  $S$ , which triggers a set  $\tau \subseteq [m]$  of base arms with feedback  $X_i^{(t)}$ 's,  $i \in \tau$
- 14:   For every  $i \in \tau$ , update  $T_i$  and  $\hat{\mu}_i$ :  $T_i = T_i + 1$ ,  $\hat{\mu}_i = \hat{\mu}_i + (X_i^{(t)} - \hat{\mu}_i)/T_i$
- 15: **end loop**

Let  $\tilde{S} = \{i \in [m] \mid p_i^S > 0\}$  be the set of arms that could be triggered by  $S$ . Let  $K = \max_{S \in \mathcal{S}} |\tilde{S}|$ . For convenience, we use  $\lceil x \rceil_0$  to denote  $\max\{\lceil x \rceil, 0\}$  for any real number  $x$ .

**Theorem 1.** For the CUCB algorithm on a CMAB- $T$  problem instance that satisfies monotonicity (Condition 1) and 1-norm TPM bounded smoothness (Condition 2) with bounded smoothness constant  $B$ ,

(1) if  $\Delta_{\min} > 0$ , we have distribution-dependent bound

$$\begin{aligned} \text{Reg}_{\mu, \alpha, \beta}(T) &\leq \sum_{i \in [m]} \frac{624B^2 K \ln T}{\Delta_{\min}^i} \\ &+ \sum_{i \in [m]} \left( \left\lceil \log_2 \frac{2BK}{\Delta_{\min}^i} \right\rceil_0 + 2 \right) \cdot \frac{\pi^2}{6} \cdot \Delta_{\max} + 6Bm; \end{aligned} \quad (1)$$

(2) we have distribution-independent bound

$$\begin{aligned} \text{Reg}_{\mu, \alpha, \beta}(T) &\leq 50B\sqrt{mKT \ln T} + 6Bm \\ &+ \left( \left\lceil \log_2 \sqrt{\frac{KT}{312m \ln T}} \right\rceil_0 + 2 \right) \cdot m \cdot \frac{\pi^2}{6} \cdot \Delta_{\max}. \end{aligned} \quad (2)$$

**Theorem 2.** For the CUCB-IS algorithm on a CMAB- $T$  problem instance that satisfies monotonicity (Condition 1) and 1-norm RTPM bounded smoothness (Condition 3),

(1) if  $\Delta_{\min} > 0$ , we have distribution-dependent bound

$$\begin{aligned} \text{Reg}_{\mu, \alpha, \beta}(T) &\leq \sum_{i \in [m]} \frac{624B^2 K \ln T}{\Delta_{\min}^i} \\ &+ \sum_{i \in [m]} \left( \left\lceil \log_2 \frac{2BK}{\Delta_{\min}^i} \right\rceil_0 + 2 \right) \cdot \frac{\pi^2}{6} \cdot \Delta_{\max} + 6Bm \\ &+ (24m^3 \ln T + m) \cdot \Delta_{\max}; \end{aligned} \quad (3)$$

(2) we have distribution-independent bound

$$\begin{aligned} \text{Reg}_{\mu, \alpha, \beta}(T) &\leq 50B\sqrt{mKT \ln T} + 6Bm \\ &+ \left( \left\lceil \log_2 \sqrt{\frac{KT}{312m \ln T}} \right\rceil_0 + 2 \right) \cdot m \cdot \frac{\pi^2}{6} \cdot \Delta_{\max} \\ &+ (24m^3 \ln T + m) \cdot \Delta_{\max}. \end{aligned} \quad (4)$$

#### 4.1. Discussions and Comparisons

We first discuss the implications of the above two theorems by comparing them with several existing results.

**Comparison with (Chen et al., 2016b) and CMAB with  $\infty$ -norm bounded smoothness conditions.** Our work is a direct adaption of the study in (Chen et al., 2016b). Comparing with (Chen et al., 2016b), we see that in both theorems above, the regret bounds are not dependent on the inverse of triggering probabilities, which is the main issue in (Chen et al., 2016b). When applied to influence maximization bandit, our result is strictly stronger than that of (Chen et al., 2016b) in two aspects: (a) we remove the factor of  $1/p^*$  by using the RTPM condition; (b) we reduce a factor of  $|E|$  and  $\sqrt{|E|}$  in the dominant terms of distribution-dependent and -independent bounds, respectively, due to our use of 1-norm instead of  $\infty$ -norm conditions.

Chen et al. (2016b) use the  $\infty$ -norm version of bounded smoothness. In the supplementary material, we provide the corresponding  $\infty$ -norm TPM bounded smoothness conditions and the regret bound results. Although 1-norm and  $\infty$ -norm conditions can be transformed to each other, the transformation would result in larger bounded smoothness constants and thus weaker regret bounds. Therefore, the results on 1-norm and  $\infty$ -norm conditions do not automatically imply each other, and one should investigate the actual application instance to decide which sets of regret bounds to use. For influence maximization, cascading, and linear bandits discussed below, they all satisfy tighter 1-norm conditions, and thus applying 1-norm based results of Theorems 1 and 2 would give better regret bounds.

**Comparison with (Wen et al., 2016) on influence maximization bandits.** Let  $G = (V, E)$  be the social graph

we consider. By Lemma 2, our Theorem 2 can be applied to the influence maximization bandit with  $B = 2|V|$ , which gives concrete  $O(\log T)$  distribution-dependent and  $O(\sqrt{T \log T})$  distribution-independent bounds for the influence maximization bandit. Wen et al. (2016) also study the influence maximization bandit, but they only provide distribution-independent regret bounds when the graph is a forest. Therefore, our result on influence maximization bandit is much more general than theirs. Even limiting our result to the forest case, our result is still better, as we now explain. When the graph is a bidirectional forest, we can show that the influence maximization bandit satisfies the 1-norm TPM bounded smoothness with bounded smoothness constant  $B = \tilde{C}$ , where  $\tilde{C}$  is the size of the largest connected component in the forest. Then we can apply Theorem 1 to obtain the distribution-independent regret bound as  $O(|E|\tilde{C}\sqrt{T \log T})$ . In contrast, their regret bound is  $O(|E|C_*\sqrt{T \log T})$ , where  $C_*$  is parameter with complicated dependency on the graph topology and edge influence probabilities. Clearly, our regret bound is  $O(\sqrt{\log T})$  better than theirs when considering asymptotic performance on  $T$ . When comparing  $\tilde{C}$  with  $C_*$ , we have  $\tilde{C} \leq |V|$  and  $C_* \leq |V|^{\frac{3}{2}}$ , thus our worst-case bound is better than their worst-case bound by an additional  $O(\sqrt{|V|})$  factor. They do not provide simple analytical properties on  $C_*$ , but on the three simple graph examples they gave, our parameter  $\tilde{C}$  is either comparable or better than  $C_*$ . Wen et al. (2016) also study a generalization of linear transformation of edge probabilities, which is orthogonal to our current discussion, and could be potentially incorporated into the general CMAB-T framework.

**Comparison with (Kveton et al., 2015c) on combinatorial cascading bandits** By Lemma 1, we can apply Theorem 1 to combinatorial conjunctive and disjunctive cascading bandits with bounded smoothness constant  $B = 1$ , achieving  $O(\sum_{\Delta_{\min}} \frac{1}{\Delta_{\min}} K \log T)$  distribution-dependent, and  $O(\sqrt{mKT \log T})$  distribution-independent regret. In contrast, besides having exactly these terms, the results in (Kveton et al., 2015c) have an extra factor of  $1/f^*$ , where  $f^* = \prod_{i \in S^*} p(i)$  for conjunctive cascades, and  $f^* = \prod_{i \in S^*} (1 - p(i))$  for disjunctive cascades, with  $S^*$  being the optimal solution and  $p(i)$  being the probability of success for item (arm)  $i$ . For conjunctive cascades,  $f^*$  could be reasonably close to 1 in practice as argued in (Kveton et al., 2015c), but for disjunctive cascades,  $f^*$  could be quite small since items in optimal solutions typically have large  $p(i)$  values. Therefore, our result completely removes the dependency on  $1/f^*$  and is better than their result. Moreover, we also have much smaller constant factors owing to a new analytical method, same as the discussion on linear bandits below.

**Comparison with (Kveton et al., 2015b) on linear bandits.** When there is no probabilistically triggered arms (i.e.  $p^* = 1$ ), Theorem 1 would have tighter bounds since some analysis dealing with probabilistic triggering is not needed. In particular, in Eq. (1) the leading constant 624 would be reduced to 48, the  $\lceil \log_2 x \rceil_0$  term is gone, and  $6Bm$  becomes  $2Bm$ ; in Eq. (4) the leading constant 50 is reduced to 14, and the other changes are the same as above (see the supplementary material). The result itself is also a new contribution, since it generalizes the linear bandit of (Kveton et al., 2015b) to general 1-norm conditions with matching regret bounds, while significantly reducing the leading constants (their constants are 534 and 47 for distribution-dependent and independent bounds, respectively). This improvement comes from a new analytical method we employ, which is briefly explained next.

## 4.2. Novel Ideas in the Proofs

Due to the space limit, the full proofs of Theorem 1 and 2 are moved to the supplementary material. Here we briefly explain the novel aspects of our analysis that allow us to achieve new regret bounds and differentiate us from previous analyses such as the ones in (Chen et al., 2016b) and (Kveton et al., 2015c;b).

We first give an intuitive explanation on how to incorporate the TPM bounded smoothness condition to remove the factor  $1/p^*$  in the regret bound. Consider a simple illustrative example of two actions  $S_0$  and  $S$ , where  $S_0$  has a fixed reward  $r_0$  as a reference action, and  $S$  has a stochastic reward depending on the outcomes of its triggered base arms. Let  $\tilde{S}$  be the set of arms that can be triggered by  $S$ . For  $i \in \tilde{S}$ , suppose  $i$  can be triggered by action  $S$  with probability  $p_i^S$ , and its true mean is  $\mu_i$  and its empirical mean at the end of round  $t$  is  $\hat{\mu}_{i,t}$ . The analysis in (Chen et al., 2016b) would need a property that, if for all  $i \in \tilde{S}$   $|\hat{\mu}_{i,t} - \mu_i| \leq \delta_i$  for some properly defined  $\delta_i$ , then  $S$  no longer generates regrets (either it is determined to be the best action or it will not be selected to play). The analysis would conclude that arm  $i$  needs to be triggered  $\Theta(\log T / \delta_i^2)$  times for the above condition to happen. Since arm  $i$  is only triggered with probability  $p_i^S$ , it means action  $S$  may need to be played  $\Theta(\log T / (p_i^S \delta_i^2))$  times. This is the essential reason why the factor of  $1/p_i^S$  or  $1/p^*$  appears in the regret bound.

Now with the TPM bounded smoothness, we know that the impact of  $|\hat{\mu}_{i,t} - \mu_i| \leq \delta_i$  to the difference in the expected reward is only  $p_i^S \delta_i$ , or equivalently, we could relax the requirement to  $|\hat{\mu}_{i,t} - \mu_i| \leq \delta_i / p_i^S$  to achieve the same effect as in the previous analysis. This translates to the result that action  $S$  would generate regret in at most  $O(\log T / (p_i^S (\delta_i / p_i^S)^2)) = O(p_i^S \log T / \delta_i^2)$  rounds.

We then need to handle the case when we have multiple actions that could trigger arm  $i$ . The simple addition of

$\sum_{S: p_i^S > 0} p_i^S \log T / \delta_i^2$  is not feasible since we may have exponentially many such actions. Instead, we introduce the key idea of *triggering probability groups*, such that the above actions are divided into groups by putting their triggering probabilities  $p_i^S$  into geometrically separated bins:  $[1, 1], (1, 1/2], (1/2, 1/4], \dots, (2^{-j+1}, 2^{-j}], \dots$ . The actions in the same group would generate regret in at most  $O(2^{-j+1} \log T / \delta_i^2)$  rounds by the above argument, and summing up together, they could generate regret in at most  $O(\sum_j 2^{-j+1} \log T / \delta_i^2) = O(\log T / \delta_i^2)$  rounds. Therefore, the factor of  $1/p_i^S$  or  $1/p^*$  is completely removed from the regret bound.

Next, we briefly explain our idea to achieve the improved bound over the linear bandit result in (Kveton et al., 2015b). The key step is to bound regret  $\Delta_{S_t}$  generated in round  $t$ . By a derivation similar to (Kveton et al., 2015b; Chen et al., 2016b) together with the 1-norm TPM bounded smoothness condition, we would obtain that  $\Delta_{S_t} \leq B \sum_{i \in \tilde{S}_t} p_i^{\mu, S_t} (\bar{\mu}_{i,t} - \mu_i)$  with high probability. The analysis in (Kveton et al., 2015b) would analyze the errors  $|\bar{\mu}_{i,t} - \mu_i|$  by a cascade of infinitely many sub-cases of whether there are  $x_j$  arms with errors larger than  $y_j$  with decreasing  $y_j$ , but it may still be loose. Instead we directly work on the above summation. Naive bounding the above error summation would not give a  $O(\log T)$  bound because there could be too many arms with small errors. Our trick is to use a *reverse amortization*: we cumulate small errors on many sufficiently sampled arms and treat them as errors of insufficiently sample arms, such that an arm sampled  $O(\log T)$  times would not contribute toward the regret. This trick tightens our analysis and leads to significantly improved constant factors.

## 5. Lower Bound of the General CMAB-T Model

In this section, we show that there exists some CMAB-T problem instance such that the regret bound in (Chen et al., 2016b) is tight, i.e. the factor  $1/p^*$  in the distribution-dependent bound and  $\sqrt{1/p^*}$  in the distribution-independent bound are unavoidable, where  $p^*$  is the minimum positive probability that any base arm  $i$  is triggered by any action  $S$ . It also implies that the TPM bounded smoothness may not be applied to all CMAB-T instances.

For our purpose, we only need a simplified version of the bounded smoothness condition of (Chen et al., 2016b) as below: There exists a bounded smoothness constant  $B$  such that, for every action  $S$  and every pair of mean outcome vectors  $\mu$  and  $\mu'$ , we have  $|r_S(\mu) - r_S(\mu')| \leq B \max_{i \in \tilde{S}} |\mu_i - \mu'_i|$ , where  $\tilde{S}$  is the set of arms that could possibly be triggered by  $S$ .

We prove the lower bounds using the following CMAB-T problem instance  $([m], \mathcal{S}, \mathcal{D}, D^{\text{trig}}, R)$ . For each base arm  $i \in [m]$ , we define an action  $S_i$ , with the set of actions  $\mathcal{S} = \{S_1, \dots, S_m\}$ . The family of distributions  $\mathcal{D}$  consists of distributions generated by every  $\mu \in [0, 1]^m$  such that the arms are independent Bernoulli variables. When playing action  $S_i$  in round  $t$ , with a fixed probability  $p$ , arm  $i$  is triggered and its outcome  $X_i^{(t)}$  is observed, and the reward of playing  $S_i$  is  $p^{-1} X_i^{(t)}$ ; otherwise with probability  $1 - p$  no arm is triggered, no feedback is observed and the reward is 0. Following the CMAB-T framework, this means that  $D^{\text{trig}}(S_i, X)$ , as a distribution on the subsets of  $[m]$ , is either  $\{i\}$  with probability  $p$  or  $\emptyset$  with probability  $1 - p$ , and the reward  $R(S_i, X, \tau) = p^{-1} X_i \cdot \mathbb{I}\{\tau = \{i\}\}$ . The expected reward  $r_{S_i}(\mu) = \mu_i$ . So this instance satisfies the above bounded smoothness with constant  $B = 1$ . We denote the above instance as FTP( $p$ ), standing for fixed triggering probability instance. For the FTP( $p$ ) instance, we have  $p^* = p$ . Then applying the result in (Chen et al., 2016b), we have distributed-dependent upper bound  $O(\sum_i \frac{1}{p \Delta_i^{\min}} \log T)$  and distribution-independent upper bound  $O\left(\sqrt{\frac{mT \log T}{p}}\right)$ .

### 5.1. Distribution-independent Lower Bound

**Theorem 3.** *Let  $p$  be a real number with  $0 < p < 1$ . Then for any CMAB-T algorithm  $A$ , if  $T \geq 6p^{-1}$ , there exists a CMAB-T environment instance  $D$  such that on instance FTP( $p$ ),*

$$\text{Reg}_D^A(T) \geq \frac{1}{170} \sqrt{\frac{mT}{p}}.$$

The proof of the above and the next theorems are all based on the results for the classical MAB problems. Comparing to the upper bound obtained from the result in (Chen et al., 2016b), we see that the upper bound is tight up to an  $O(\sqrt{\log T})$  factor.

### 5.2. Distribution-dependent Lower Bound

For a learning algorithm, we say that it is *consistent* if, for every  $\mu$ , every non-optimal arm is played  $o(T^a)$  times in expectation, for any real number  $a > 0$ . Then we have the following distribution-dependent lower bound.

**Theorem 4.** *For any consistent algorithm  $A$  running on instance FTP( $p$ ) and  $\mu_i < 1$  for every arm  $i$ , we have*

$$\liminf_{T \rightarrow +\infty} \frac{\text{Reg}_\mu^A(T)}{\ln T} \geq \sum_{i: \mu_i < \mu^*} \frac{p^{-1} \Delta_i}{\text{kl}(\mu_i, \mu^*)},$$

where  $\mu^* = \max_i \mu_i$ ,  $\Delta_i = \mu^* - \mu_i$ , and  $\text{kl}(\cdot, \cdot)$  is the Kullback-Leibler divergence function.



Again we see that the distribution-dependent upper bound obtained from (Chen et al., 2016b) asymptotically match the lower bound above.

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## Supplementary Materials

### A. Model Discussions

#### A.1. Comparison with the framework of (Chen et al., 2016b)

The CMAB-T framework described above essentially follows the framework of (Chen et al., 2016b), but with the following noticeable differences. First, we refer to  $S$  as an abstract action from an action space  $\mathcal{S}$ , while in (Chen et al., 2016b),  $S$  is referred to as a super arm, which is a subset of base arms  $[m]$ . In the case of CMAB without probabilistically triggered arms, we can simply let every super arm  $S$  be an action, and  $\tau(S, X) = S$ , meaning that playing super arm  $S$  deterministically triggers all and only base arms in  $S \subseteq [m]$ . Second, we explicitly allows action space to be infinite or even continuous space, while in (Chen et al., 2016b), the action space is the subsets of base arms and thus is finite. We will see later that the infinite action space does not make essential difference in the analysis. Third, for probabilistically triggered arms, we explicitly use  $\tau(S, X)$  to model them, and allows  $\tau(S, X)$  to have additional randomness besides the randomness of  $X$ . In (Chen et al., 2016b), probabilistic triggering is explained as further base arms being triggered based on the outcomes of previously triggered base arms, and to model certain triggering structure or additional randomness in triggering an arm, dummy base arms need to be added. However, this may require introducing a large number of dummy base arms. For example, for the cascading bandits, to specify the order of the cascade sequence, we need to add dummy base arms corresponding to every possible order of the base arms. Moreover,  $\tau(S, X)$  cleanly separates the randomness known to the player from the unknown randomness from the environment outcome. For example, in the discount-based continuous influence maximization (Yang et al., 2016),  $\tau(c, X)$  includes the randomness of activating the seed set from the discount vector  $c$  given by  $\eta_i$ 's, which are known to the player. In contrast, the distribution of  $X_{(u,v)}$ , namely probability  $p(u, v)$  on edges are unknown and need to be learned. In this case, if we use dummy base arms to model such additional triggering behavior from marketing actions to seed sets, these dummy base arms will be mixed together with edge base arms for which the learning algorithm need to learn, unless further distinction is made.

Therefore, we believe that our current adaptation CMAB-T provides a cleaner framework and is more easily to be applied to various problem instances. We remark that all the analysis and results in (Chen et al., 2016b) remain unchanged with our current adaptation.

#### A.2. Modeling general marketing actions in influence maximization

Note that we can also use randomized  $\tau(S, X)$  to model some extended versions of influence maximization. For example, general marketing actions are proposed in (Kempe et al., 2003) and continuous discount actions are proposed in (Yang et al., 2016), both allowing activating seed nodes with a probability depending on the marketing intensity on the node. In particular, an action in the discount-based continuous influence maximization in (Yang et al., 2016) is a vector  $c = (c_1, c_2, \dots, c_n)$ , where  $c_i \in [0, 1]$  is the discount to be given to node  $i$ . Discount  $c_i$  is translated to probability  $\eta_i(c_i)$  that node  $i$  is activated as a seed, where  $\eta_i(\cdot)$  is a monotonically non-decreasing function with  $\eta_i(0) = 0$  and  $\eta_i(1) = 1$ . In this case, the probabilistic triggering function  $\tau(c, X)$  includes the randomness from  $c$  to seed activations based on  $\eta_i$ 's, beyond the randomness of  $X$ . That is, even when  $c$  and  $X$  are fixed,  $\tau(c, X)$  is still a random set. We further remark that in this case, the action space of all discount vectors is a continuous and infinite space, which is allowed in our adapted CMAB-T model.

## B. Omitted Proofs in Section 3 (On 1-Norm TPM Conditions)

### B.1. Proof of Lemma 1

*Proof.* Let  $S$  be an action. We regard  $S$  as a permutation of  $k$  of the arms. Without loss of generality, we may assume  $S = (1, \dots, k)$  for some  $k \leq K$ . For an arm  $i > k$ ,  $i$  will not be triggered by action  $S$ , and thus  $p_i^{\mu, S} = 0$ . The reward also does not depend on those arms. So we may only consider the arms  $1, \dots, k$ . For convenience, we only list the expectations of arms in  $S$ , so that  $\mu = (\mu_1, \dots, \mu_k)$  and  $\mu' = (\mu'_1, \dots, \mu'_k)$ .

Informally speaking, we can change the expectation of the arms from  $\mu_i$  to  $\mu'_i$ , in the reverse order from  $k$  to 1. Changing the expectation of an arm  $j$  does not affect the triggering probability of an arm  $i$  ordered in front of  $j$ , i.e.  $i < j$ . And when changing an arm from  $\mu_i$  to  $\mu'_i$ , the reward changes by at most  $p_i^{\mu, S} |\mu_i - \mu'_i|$ . Therefore the total difference of reward is at most  $\sum_{i=1}^k p_i^{\mu, S} |\mu_i - \mu'_i|$ .

Formally, for the conjunctive cascading bandit,  $r_S(\boldsymbol{\mu}) = \prod_{j=1}^k \mu_j$ , and  $p_i^{\boldsymbol{\mu}, S} = \prod_{j=1}^{i-1} \mu_j$  for  $i = 1, \dots, k$ . For every  $j = 0, 1, \dots, k$ , let

$$\boldsymbol{\mu}^{(j)} = (\mu_1, \dots, \mu_j, \mu'_{j+1}, \dots, \mu'_k),$$

specifically,  $\boldsymbol{\mu}^{(k)} = \boldsymbol{\mu}$ ,  $\boldsymbol{\mu}^{(0)} = \boldsymbol{\mu}'$ . Then,

$$\begin{aligned} |r_S(\boldsymbol{\mu}^{(j)}) - r_S(\boldsymbol{\mu}^{(j-1)})| &= \left| \prod_{i=1}^k \mu_i^{(j)} - \prod_{i=1}^k \mu_i^{(j-1)} \right| \\ &= \prod_{i, i \neq j} \mu_i^{(j)} \left| \mu_j^{(j)} - \mu_j^{(j-1)} \right| \\ &\leq \prod_{i=1}^{j-1} \mu_i^{(j)} \left| \mu_j^{(j)} - \mu_j^{(j-1)} \right| \\ &= \prod_{i=1}^{j-1} \mu_i \left| \mu_j - \mu'_j \right| \\ &= p_j^{\boldsymbol{\mu}, S} \left| \mu_j - \mu'_j \right|, \end{aligned}$$

$$\begin{aligned} |r_S(\boldsymbol{\mu}) - r_S(\boldsymbol{\mu}')| &= |r_S(\boldsymbol{\mu}^{(k)}) - r_S(\boldsymbol{\mu}^{(0)})| \\ &\leq \sum_{j=1}^k |r_S(\boldsymbol{\mu}^{(j)}) - r_S(\boldsymbol{\mu}^{(j-1)})| \\ &\leq \sum_{j=1}^k p_j^{\boldsymbol{\mu}, S} \left| \mu_j - \mu'_j \right|. \end{aligned}$$

For the disjunctive case, let  $\lambda_i = 1 - \mu_i$  for  $i \in [m]$ . Then we have  $r_S(\boldsymbol{\mu}) = 1 - \prod_{j=1}^k \lambda_j$ , and  $p_i^{\boldsymbol{\mu}, S} = \prod_{j=1}^{i-1} \lambda_j$ . The rest analysis follows the same pattern as the conjunctive case.  $\square$

## B.2. Proof of Lemma 2

Instead of merely proving Lemma 2 for the specific influence maximization bandit, we first introduce the following two general conditions and show that they together imply the RTPM bounded smoothness, then we show that influence maximization bandit satisfies these two conditions. Therefore, these two conditions could be of independent interest for other applications in proving the RPTM bounded smoothness condition.

Since the following condition would also be used for the general  $\infty$ -norm TPM conditions, here we also use a general version with a generic bounded smoothness function  $g$ . We use the term *bounded smoothness function*  $g$  to refer to a continuous, strictly increasing function  $g(\cdot)$  with  $g(0) = 0$ . Note that the term here does not mean that  $g$  has the bounded smoothness property, but rather it means that  $g$  would be used in the context of bounded smoothness conditions for CMAB problem instances.

**Condition 4. (Local TPM bounded smoothness)** We say that a CMAB- $T$  problem instance satisfies Local TPM bounded smoothness with a local bounded smoothness function  $g(x)$ , if for any two distributions  $D, D' \in \mathcal{D}$  with expectation vectors  $\boldsymbol{\mu}$  and  $\boldsymbol{\mu}'$ , any  $i \in [m]$ , any action  $S$ , and any  $\Lambda > 0$ , we have  $|r_S(\boldsymbol{\mu}) - r_S(\boldsymbol{\mu}')| \leq g(\Lambda)$  if  $p_i^{\boldsymbol{\mu}, S} |\mu_i - \mu'_i| \leq \Lambda$  and  $\mu_j = \mu'_j$  for  $j \neq i$ .

**Condition 5. (Local moderated continuity on triggering probability)** We say that a CMAB- $T$  problem instance satisfies local moderated continuity on triggering probability, if for any two distributions  $D, D' \in \mathcal{D}$  with expectation vectors  $\boldsymbol{\mu}$  and  $\boldsymbol{\mu}'$ , any  $i \in [m]$ , any action  $S$ , and any  $\Lambda > 0$ , we have  $|p_k^{\boldsymbol{\mu}, S} - p_k^{\boldsymbol{\mu}', S}| \leq \Lambda$  for every  $k \in [m]$  if  $p_i^{\boldsymbol{\mu}, S} |\mu_i - \mu'_i| \leq \Lambda$  and  $\mu_j = \mu'_j$  for  $j \neq i$ .

The following lemma shows that Conditions 4 and 5 together imply the 1-norm RTPM bounded smoothness (Condition 3).



**Lemma 3.** *If a CMAB-T problem instance satisfies local TPM bounded smoothness (Condition 4) with a linear local bounded smoothness function  $g(x) = Cx$  and local moderated continuity on triggering probability (Condition 5), then it satisfies the 1-norm RTPM bounded smoothness (Condition 3) with bounded smoothness constant  $B = 2C$ .*

*Proof.* Fix an arbitrary action  $S$  and an arbitrary expectation vector  $\mu$ . Without loss of generality, we may assume  $p_i^{\mu,S} \geq p_{i+1}^{\mu,S}$  for  $i = 1, \dots, m-1$ . Intuitively, we change the arms from  $m$  to 1, i.e. by non-decreasing order of triggering probability. By Condition 5, changing arms with smaller triggering probability will not significantly affect the large triggering probabilities. More precisely, we can show inductively that the triggering probability of arm  $i$  at most increases to  $2p_i^{\mu,S}$  by changing the arms “smaller” than  $i$  by at most  $\frac{1}{2m}$  each. Then we can show the difference of reward is at most  $m \cdot g(2\Lambda)$  by adding the difference caused by the change of each arm.

Formally, consider any  $\mu' = (\mu'_1, \mu'_2, \dots, \mu'_m)$  and any  $\Lambda > 0$  with  $|\mu_i - \mu'_i| \leq \frac{1}{2m}$  and  $\max_i p_i^{\mu,S} |\mu_i - \mu'_i| \leq \Lambda$ , for all  $i \in [m]$ . Let

$$\mu^{(j)} = (\mu_1, \dots, \mu_j, \mu'_{j+1}, \dots, \mu'_m),$$

for every  $j$ , specifically,  $\mu^{(m)} = \mu$ ,  $\mu^{(0)} = \mu'$ . Then we first prove  $p_j^{\mu^{(j)},S} \leq 2p_j^{\mu,S}$  by induction on  $j$  from  $m$  back to 1. The base case of  $j = m$  is trivial. For every  $1 \leq j < m$ , by Condition 5,

$$\begin{aligned} \left| p_j^{\mu^{(j)},S} - p_j^{\mu^{(j-1)},S} \right| &\leq p_i^{\mu^{(j-1)},S} |\mu_i - \mu'_i| \\ &\leq \frac{p_i^{\mu^{(j-1)},S}}{2m} \\ &\leq \frac{2p_i^{\mu,S}}{2m} \\ &\leq \frac{p_j^{\mu,S}}{m}. \end{aligned}$$

$$\begin{aligned} p_j^{\mu^{(j)},S} &\leq p_j^{\mu,S} + \sum_{i=j+1}^m \left| p_j^{\mu^{(i-1)},S} - p_j^{\mu^{(i)},S} \right| \\ &\leq p_j^{\mu,S} + \sum_{i=j+1}^m \frac{p_j^{\mu,S}}{m} \\ &\leq p_j^{\mu,S} + \frac{mp_j^{\mu,S}}{m} \\ &= 2p_j^{\mu,S}. \end{aligned}$$

Thus, the induction step is also correct. Then by Condition 4, we have

$$\begin{aligned} |r_S(\mu) - r_S(\mu')| &\leq \sum_{j=1}^m \left| r_S(\mu^{(j)}) - r_S(\mu^{(j-1)}) \right| \\ &\leq \sum_{j=1}^m g(p_j^{\mu^{(j)},S} |\mu_j - \mu'_j|) \\ &\leq \sum_{j=1}^m g(2p_j^{\mu,S} |\mu_j - \mu'_j|) \\ &= 2C \sum_{j=1}^m p_j^{\mu,S} |\mu_j - \mu'_j|. \end{aligned} \tag{5}$$

*Proof of Lemma 2.* We show that online influence maximization problem satisfies local TPM bounded smoothness (Condition 4) with  $g(x) = |V|x$  and local moderated continuity on triggering probability (Condition 5), then the lemma holds following Lemma 3. □

Let  $m = |E|$  be the number of arms. For an edge  $e$ , we also use  $e$  to denote the corresponding arm. Fixing an action  $S$ , which is a seed set, let  $\mathcal{E}_v$  be the event that node  $v$  is activated. The occurrence of  $\mathcal{E}_v$  is equivalent to that there is a path from some node in  $S$  to  $v$  where each edge  $e$  on the path has  $X_e = 1$ . Fixing an edge  $e$ , consider two expectation vectors  $\mu$  and  $\mu'$  that  $\mu_{e'} = \mu'_{e'}$  for every edge  $e'$  except  $e$ . The probability that  $e$  is triggered in a run,  $p_e^{\mu, S}$  does not depend on  $\mu_e$ . And  $\Pr\{\mathcal{E}_v \mid e \text{ is not triggered}\}$  also does not depend on  $\mu_e$ . If  $e$  is triggered, the difference of  $\Pr\{\mathcal{E}_v \mid e \text{ is triggered}\}$  is at most  $|\mu_e - \mu'_e|$  when  $\mu_e$  is changed to  $\mu'_e$ . We use  $\Pr_\mu$  and  $\Pr_{\mu'}$  to denote the probability of an event with respect to  $\mu$  and  $\mu'$ , then  $|\Pr_\mu\{\mathcal{E}_v\} - \Pr_{\mu'}\{\mathcal{E}_v\}| \leq p_e^{\mu, S} |\mu_e - \mu'_e|$ . Since the probability that an edge  $e' = (v, u)$  is triggered is just  $\Pr\{\mathcal{E}_v\}$ , influence maximization bandit satisfies local moderated continuity on triggering probability. Moreover, by definition  $r_S(\mu) = \sum_{v \in V} \Pr_\mu\{\mathcal{E}_v\}$ . Thus we have

$$\begin{aligned} |r_S(\mu) - r_S(\mu')| &= \left| \sum_{v \in V} \Pr_\mu\{\mathcal{E}_v\} - \sum_{v \in V} \Pr_{\mu'}\{\mathcal{E}_v\} \right| \\ &\leq \sum_{v \in V} |\Pr_\mu\{\mathcal{E}_v\} - \Pr_{\mu'}\{\mathcal{E}_v\}| \\ &\leq \sum_{v \in V} p_e^{\mu, S} |\mu_e - \mu'_e| \\ &= |V| p_e^{\mu, S} |\mu_e - \mu'_e|. \end{aligned}$$

Thus the influence maximization bandit satisfies the local TPM bounded smoothness with  $g(x) = |V|x$ .  $\square$

## C. Proofs in Section 4

### C.1. Basics of CMAB-T problems

We utilize the following well known tail bound in our analysis.

**Fact 1 (Hoeffding's Inequality (Hoeffding, 1963)).** *Let  $X_1, \dots, X_n$  be independent and identically distributed random variables with common support  $[0, 1]$  and mean  $\mu$ . Let  $Y = X_1 + \dots + X_n$ . Then for all  $\delta \geq 0$ ,*

$$\Pr\{|Y - n\mu| \geq \delta\} \leq 2e^{-2\delta^2/n}.$$

**Fact 2 (Multiplicative Chernoff Bound (Mitzenmacher & Upfal, 2005)).** <sup>1</sup> *Let  $X_1, \dots, X_n$  be Bernoulli random variables taking values from  $\{0, 1\}$ , and  $\mathbb{E}[X_t | X_1, \dots, X_{t-1}] \geq \mu$  for every  $t \leq n$ . Let  $Y = X_1 + \dots + X_n$ . Then for all  $0 < \delta < 1$ ,*

$$\Pr\{Y \leq (1 - \delta)n\mu\} \leq e^{-\frac{\delta^2 n\mu}{2}}.$$

We introduce the following definition to assist our analysis.

**Definition 3 (Event-Filtered Regret).** *For any series of events  $\{\mathcal{E}_t\}_{t \geq 1}$  indexed by round number  $t$ , we define  $\text{Reg}_{\mu, \alpha}^A(T, \{\mathcal{E}_t\}_{t \geq 1})$  as the regret filtered by events  $\{\mathcal{E}_t\}_{t \geq 1}$ , that is, regret is only counted in round  $t$  if  $\mathcal{E}_t$  happens in round  $t$ . Formally,*

$$\text{Reg}_{\mu, \alpha}^A(T, \{\mathcal{E}_t\}_{t \geq 1}) = \mathbb{E} \left[ \sum_{t=1}^T \mathbb{I}(\mathcal{E}_t) (\alpha \cdot \text{opt}_\mu - r_\mu(S_t^A)) \right].$$

For convenience,  $A$ ,  $\alpha$ ,  $\mu$  and/or  $T$  can be omitted when the context is clear, and we simply use  $\text{Reg}_{\mu, \alpha}^A(T, \mathcal{E}_t)$   $\text{Reg}_{\mu, \alpha}^A(T, \{\mathcal{E}_t\}_{t \geq 1})$ .

The following definition means we are not so unlucky that  $\hat{\mu}_{i, t-1}$  is not as accurate as expected.

**Definition 4.** *We say that the sampling is nice at the beginning of round  $t$  if for every arm  $i \in [m]$ ,  $|\hat{\mu}_{i, t-1} - \mu_i| < \rho_{i, t}$ , where  $\rho_{i, t} = \sqrt{\frac{3 \ln t}{2T_{i, t-1}}}$  in round  $t$ . Let  $\mathcal{N}_t^s$  be such event.*

<sup>1</sup>The result in the book by (Mitzenmacher & Upfal, 2005) (Theorem 4.5 together with Exercise 4.7) only covers the case where random variables  $X_i$ 's are independent. However the result can be easily generalized to our case with an almost identical proof. The only main change is to replace  $\mathbb{E} \left[ e^{t(\sum_{j=1}^{i-1} X_j + X_i)} \right] = \mathbb{E} \left[ e^{t \sum_{j=1}^{i-1} X_j} \right] \mathbb{E} \left[ e^{t X_i} \right]$  with  $\mathbb{E} \left[ e^{t(\sum_{j=1}^{i-1} X_j + X_i)} \right] = \mathbb{E} \left[ e^{t \sum_{j=1}^{i-1} X_j} \mathbb{E} \left[ e^{t X_i} \mid X_1, \dots, X_{i-1} \right] \right]$ .

**Lemma 4.** For each round  $t \geq 1$ ,  $\Pr\{\neg \mathcal{N}_t^s\} \leq 2mt^{-2}$ .

*Proof.* For each round  $t \geq 1$ , we have

$$\begin{aligned} \Pr\{\neg \mathcal{N}_t^s\} &= \Pr\left\{\exists i \in [m], |\hat{\mu}_{i,t-1} - \mu_i| \geq \sqrt{\frac{3 \ln t}{2T_{i,t-1}}}\right\} \\ &\leq \sum_{i \in [m]} \Pr\left\{|\hat{\mu}_{i,t-1} - \mu_i| \geq \sqrt{\frac{3 \ln t}{2T_{i,t-1}}}\right\} \\ &= \sum_{i \in [m]} \sum_{k=1}^{t-1} \Pr\left\{T_{i,t-1} = k, |\hat{\mu}_{i,t-1} - \mu_i| \geq \sqrt{\frac{3 \ln t}{2T_{i,t-1}}}\right\}. \end{aligned} \quad (6)$$

When  $T_{i,t-1} = k$ ,  $\hat{\mu}_{i,t-1}$  is the average of  $k$  i.i.d. random variables  $X_i^{[1]}, \dots, X_i^{[k]}$ , where  $X_i^{[j]}$  is the outcome of arm  $i$  when it is triggered for the  $j$ -th time during the execution. That is,  $\hat{\mu}_{i,t-1} = \sum_{j=1}^k X_i^{[j]} / k$ . Then we have

$$\begin{aligned} \Pr\left\{T_{i,t-1} = k, |\hat{\mu}_{i,t-1} - \mu_i| \geq \sqrt{\frac{3 \ln t}{2T_{i,t-1}}}\right\} &= \Pr\left\{T_{i,t-1} = k, \left|\sum_{j=1}^k X_i^{[j]} / k - \mu_i\right| \geq \sqrt{\frac{3 \ln t}{2k}}\right\} \\ &\leq \Pr\left\{\left|\sum_{j=1}^k X_i^{[j]} - k\mu_i\right| \geq \sqrt{\frac{3k \ln t}{2}}\right\} \leq 2t^{-3}, \end{aligned} \quad (7)$$

where the last inequality uses the Hoeffding's Inequality (Fact 1). Combining Inequalities (6) and (7), we thus prove the lemma.  $\square$

**Definition 5 (Triggering probability (TP) group).** Let  $i$  be an arm and  $j$  be a natural number, define the triggering probability group (of actions)

$$\mathcal{S}_{i,j}^\mu = \{S \in \mathcal{S} \mid 2^{-j} \leq p_i^{\mu,S} < 2^{-j+1}\}.$$

For  $j = 0$ , there is an alternative definition as below:

$$\mathcal{S}_{i,0}^\mu = \{S \in \mathcal{S} \mid p_i^{\mu,S} = 1\}.$$

$\{\mathcal{S}_{i,j}^\mu\}_{j \geq 0}$  forms a partition of  $\{S \in \mathcal{S} \mid p_i^{\mu,S} > 0\}$ .

**Definition 6 (Counter).** For each TP group  $\mathcal{S}_{i,j}$ , we define a corresponding counter  $N_{i,j}$ . In a run of a learning algorithm, the counters are maintained in the following manner. All the counters are initialized to 0. In each round  $t$ , if the action  $S_t$  is chosen, then update  $N_{i,j}$  to  $N_{i,j} + 1$  for every  $(i, j)$  that  $S_t \in \mathcal{S}_{i,j}^\mu$ . Denote  $N_{i,j}$  at the end of round  $t$  with  $N_{i,j,t}$ . In other words, we can define the counters with the recursive equation below:

$$N_{i,j,t} = \begin{cases} 0, & \text{if } t = 0 \\ N_{i,j,t-1} + 1, & \text{if } t > 0, S_t \in \mathcal{S}_{i,j}^\mu \\ N_{i,j,t-1}, & \text{otherwise.} \end{cases}$$

**Definition 7.** We say that the triggering is nice at the beginning of round  $t$  (with respect to  $j_{\max}^i$ ), if for every TP group (Definition 5) identified by arm  $i$  and  $1 \leq j \leq j_{\max}^i$ , if  $\sqrt{\frac{6 \ln t}{\frac{1}{3}N_{i,j,t-1} \cdot 2^{-j}}} \leq 1$ , then  $T_{i,t-1} \geq \frac{1}{3}N_{i,j,t-1} \cdot 2^{-j}$ . We denote this event with  $\mathcal{N}_t^i$ . It implies

$$\rho_{i,t} = \sqrt{\frac{3 \ln t}{2T_{i,t-1}}} \leq \sqrt{\frac{3 \ln t}{2 \cdot \frac{1}{3}N_{i,j,t-1} \cdot 2^{-j}}}.$$

**Lemma 5.** For a series of integers  $\{j_{\max}^i\}_{i \in [m]}$ ,  $\Pr\{\neg \mathcal{N}_t^i\} \leq \sum_{i \in [m]} j_{\max}^i t^{-2}$  for every round  $t \geq 1$ .

*Proof.* We prove this lemma by showing  $\Pr\{N_{i,j,t-1} = s, T_{i,t-1} \leq \frac{1}{3}N_{i,j,t-1} \cdot 2^{-j}\} \leq t^{-3}$ , for  $0 \leq s \leq t-1$  and  $\sqrt{\frac{6 \ln t}{s \cdot 2^{-j}}} \leq 1$ . Let  $t_k$  be the round that  $N_{i,j}$  is increased for the  $k$ -th time, for  $1 \leq k \leq s$ . Let  $Y_k = \mathbb{I}\{i \in \tau_{t_k}\}$  be a

Bernoulli variable, that is,  $i$  is triggered in round  $t_k$ . When fixing the action  $S_{t_k}$ ,  $Y_k$  is independent from  $Y_1, \dots, Y_{k-1}$ . Since  $S_{t_k} \in \mathcal{S}_{i,j}$ ,  $\mathbb{E}[Y_k \mid Y_1, \dots, Y_{k-1}] \geq 2^{-j}$ . Let  $Z = Y_1 + \dots + Y_s$ . By multiplicative Chernoff bound (Fact 2), we have

$$\Pr \left\{ Z < \frac{1}{3}s \cdot 2^{-j} \right\} < \exp \left( - \left( \frac{2}{3} \right)^2 18 \ln t / 2 \right) < \exp(-3 \ln t) = t^{-3}.$$

By definition of  $T_i$ , there is  $T_{i,t-1} \geq Z$ . So  $\Pr\{N_{i,j,t-1} = s, T_{i,t-1} \leq \frac{1}{3}N_{i,j,t-1} \cdot 2^{-j}\} \leq t^{-3}$ . By taking  $i$  over  $[m]$ ,  $j$  over  $1, \dots, j_{\max}^i$ ,  $s$  over  $0, \dots, t-1$ , the lemma holds.  $\square$

**Lemma 6 (Bound of intentional sampling regret for CUCB-IS).** *For the CUCB-IS algorithm on any problem instance,*

$$\text{Reg}(\{\exists i, \rho_{i,t} \geq \frac{1}{4m}\}) \leq (24m^3 \ln T + m) \cdot \Delta_{\max}.$$

*Proof.* According to CUCB-IS algorithm, if there is such an arm  $i$  that  $\rho_{i,t} \geq \frac{1}{4m}$  in round  $t$ , it will choose an action  $S$  that  $p_i^S = 1$  for one of such arms. For each arm  $i$ ,  $\rho_{i,t} \geq \frac{1}{4m}$  implies

$$T_{i,t-1} \leq 24m^2 \ln t \leq 24m^2 \ln T.$$

So if  $T_i$  has been greater than  $24m^2 \ln T$ , arm  $i$  will not be intentionally sampled. Then each arm is intentionally sampled for at most  $24m^2 \ln T + 1$  times and the regret is no more than  $(24m^3 \ln T + m) \cdot \Delta_{\max}$ .  $\square$

## C.2. The Case of No Probabilistically Triggered Arms

In this section, we state and prove a theorem for the case of no probabilistically triggered arms, i.e.  $p^* = 1$ , when the CMAB-T instance satisfies the 1-norm (non-TPM) bounded smoothness condition below.

**Condition 6 (1-Norm Bounded Smoothness).** *We say that a CMAB-T problem instance satisfies 1-norm bounded smoothness, if there exists a bounded smoothness constant  $B \in \mathbb{R}^+$  such that, for any two distributions  $D, D' \in \mathcal{D}$  with expectation vectors  $\mu$  and  $\mu'$ , and any action  $S$ , we have  $|r_S(\mu) - r_S(\mu')| \leq B \sum_{i \in \tilde{S}} |\mu_i - \mu'_i|$ , where  $\tilde{S}$  is the set of arms that are triggered by  $S$ .*

As discussed in the main text, this theorem provides better bounds than Theorem 1 with probabilistically triggered arms. Its proof is also simpler, so the readers could choose to either get oneself familiar with the analysis with this proof first, or directly just to the next section for the proof of Theorem 1.

**Theorem 5.** *For the CUCB algorithm on a CMAB (without triggering, i.e.  $p^* = 1$ ) problem that satisfies 1-norm bounded smoothness (Condition 6) with bounded smoothness constant  $B$ ,*

1. if  $\Delta_{\min} > 0$ , we have distribution-dependent bound

$$\begin{aligned} \text{Reg}_{\mu, \alpha, \beta}(T) &\leq \sum_{i \in [m]} \frac{48B^2 K \ln T}{\Delta_{\min}^i} + 2Bm \\ &\quad + \frac{\pi^2}{3} \cdot m \cdot \Delta_{\max}; \end{aligned} \tag{8}$$

2. we have distribution-independent bound

$$\begin{aligned} \text{Reg}_{\mu, \alpha, \beta}(T) &\leq 14B\sqrt{KmT \ln T} + 2Bm \\ &\quad + \frac{\pi^2}{3} \cdot m \cdot \Delta_{\max}; \end{aligned} \tag{9}$$

*Proof of Theorem 5.* To unify the proofs for distribution-dependent and distribution-independent bounds, we introduce a positive real number  $M_i$  for each arm  $i$ . Let  $\mathcal{F}_t$  be the event  $\{r_{S_t}(\bar{\mu}) < \alpha \cdot \text{opt}(\bar{\mu})\}$ . In other words,  $\mathcal{F}_t$  means the oracle fails in round  $t$ . By assumption,  $\Pr\{\mathcal{F}_t\} \leq 1 - \beta$ . Define  $M_S = \max_{i \in \tilde{S}} M_i$  for each action  $S$ , specifically,  $M_S = 0$  if  $\tilde{S} = \emptyset$ . Define

$$\kappa_T(M, s) = \begin{cases} 2B, & \text{if } s = 0, \\ 2B\sqrt{\frac{6 \ln T}{s}}, & \text{if } 1 \leq s \leq \ell_T(M), \\ 0, & \text{if } s \geq \ell_T(M) + 1, \end{cases}$$



where

$$\ell_T(M) = \left\lfloor \frac{24B^2K^2 \ln T}{M^2} \right\rfloor.$$

We then show that if  $\{\Delta_{S_t} \geq M_{S_t}\}$ ,  $\neg \mathcal{F}_t$  and  $\mathcal{N}_t^s$  hold, we have

$$\Delta_{S_t} \leq \sum_{i \in \tilde{S}_t} \kappa_T(M_i, T_{i,t-1}). \quad (10)$$

The right hand side of the inequality is non-negative, so it holds naturally if  $\Delta_{S_t} = 0$ . We only need to consider  $\Delta_{S_t} > 0$ . By  $\mathcal{N}_t^s$  and  $\neg \mathcal{F}_t$ , we have

$$r_{S_t}(\bar{\mu}_t) \geq \alpha \cdot \text{opt}(\bar{\mu}_t) \geq \alpha \cdot \text{opt}(\mu) = r_{S_t}(\mu) + \Delta_{S_t},$$

Then by Condition 2,

$$\Delta_{S_t} \leq r_{S_t}(\bar{\mu}_t) - r_{S_t}(\mu) \leq B \sum_{i \in \tilde{S}_t} (\bar{\mu}_{i,t} - \mu_i).$$

We are going to bound  $\Delta_{S_t}$  by bounding  $\bar{\mu}_{i,t} - \mu_i$ . But before doing so, we first perform a transformation. As we have  $\Delta_{S_t} \geq M_{S_t}$ , so  $B \sum_{i \in \tilde{S}_t} (\bar{\mu}_{i,t} - \mu_i) \geq \Delta_{S_t} \geq M_{S_t}$ . We have

$$\begin{aligned} \Delta_{S_t} &\leq B \sum_{i \in \tilde{S}_t} (\bar{\mu}_{i,t} - \mu_i) \\ &\leq -M_{S_t} + 2B \sum_{i \in \tilde{S}_t} (\bar{\mu}_{i,t} - \mu_i) \\ &= 2B \sum_{i \in \tilde{S}_t} \left[ (\bar{\mu}_{i,t} - \mu_i) - \frac{M_{S_t}}{2B|\tilde{S}_t|} \right] \\ &\leq 2B \sum_{i \in \tilde{S}_t} \left[ (\bar{\mu}_{i,t} - \mu_i) - \frac{M_{S_t}}{2BK} \right] \\ &\leq 2B \sum_{i \in \tilde{S}_t} \left[ (\bar{\mu}_{i,t} - \mu_i) - \frac{M_i}{2BK} \right]. \end{aligned} \quad (11)$$

By  $\mathcal{N}_t^s$ , we have  $\bar{\mu}_{i,t} - \mu_i \leq \min\{2\rho_{i,t}, 1\}$ . So

$$(\bar{\mu}_{i,t} - \mu_i) - \frac{M_i}{2BK} \leq \min\{2\rho_{i,t}, 1\} - \frac{M_i}{2BK} \leq \min\left\{2\sqrt{\frac{3 \ln T}{2T_{i,t-1}}}, 1\right\} - \frac{M_i}{2BK}.$$

If  $T_{i,t-1} \leq \ell_T(M_i)$ , we have  $(\bar{\mu}_{i,t} - \mu_i) - \frac{M_i}{2BK} \leq \min\left\{2\sqrt{\frac{3 \ln T}{2T_{i,t-1}}}, 1\right\} \leq \frac{1}{2B} \kappa_T(M_i, T_{i,t-1})$ . If  $T_{i,t-1} \geq \ell_T(M_i) + 1$ , then  $2\sqrt{\frac{3 \ln T}{2T_{i,t-1}}} \leq \frac{M_i}{2BK}$ , so  $(\bar{\mu}_{i,t} - \mu_i) - \frac{M_i}{2BK} \leq 0 = \frac{1}{2B} \kappa_T(M_i, T_{i,t-1})$ . In conclusion, we continue (11) with

$$(11) \leq \sum_{i \in \tilde{S}_t} \kappa_T(M_i, T_{i,t-1}).$$

Then in each run,

$$\begin{aligned}
 \sum_{t=1}^T \mathbb{I}(\{\Delta_{S_t} \geq M_{S_t}\} \wedge \neg \mathcal{F}_t \wedge \mathcal{N}_t^s) \cdot \Delta_{S_t} &\leq \sum_{t=1}^T \sum_{i \in S_t} \kappa_T(M_i, T_{i,t-1}) \\
 &= \sum_{i \in [m]} \sum_{s=0}^{T_{i,T}} \kappa_T(M_i, s) \\
 &\leq \sum_{i \in [m]} \sum_{s=0}^{\ell_T(M_i)} \kappa_T(M_i, s) \\
 &= 2Bm + \sum_{i \in [m]} \sum_{s=1}^{\ell_T(M_i)} 2B \sqrt{\frac{6 \ln T}{s}} \\
 &\leq 2Bm + \sum_{i \in [m]} \int_{s=0}^{\ell_T(M_i)} 2B \sqrt{\frac{6 \ln T}{s}} ds \\
 &\leq 2Bm + \sum_{i \in [m]} 4B \sqrt{6 \ln T \ell_T(M_i)} \\
 &\leq 2Bm + \sum_{i \in [m]} 4B \sqrt{6 \ln T \cdot \frac{24B^2 K^2 \ln T}{M_i^2}} \\
 &\leq 2Bm + \sum_{i \in [m]} \frac{48B^2 K \ln T}{M_i}.
 \end{aligned}$$

So

$$\text{Reg}(\{\Delta_{S_t} \geq M_{S_t}\} \wedge \neg \mathcal{F}_t \wedge \mathcal{N}_t^s) = \mathbb{E} \left[ \sum_{t=1}^T \mathbb{I}(\{\Delta_{S_t} \geq M_{S_t}\} \wedge \neg \mathcal{F}_t \wedge \mathcal{N}_t^s) \cdot \Delta_{S_t} \right] \leq 2Bm + \sum_{i \in [m]} \frac{48B^2 K \ln T}{M_i}.$$

By Lemma 4,  $\Pr\{\neg \mathcal{N}_t^s\} \leq 2t^{-2}$ . Then, as  $\text{Reg}(\mathcal{E}_t) \leq \sum_{t=1}^T \Pr\{\mathcal{E}_t\} \Delta_{\max}$  by definition of filtered regret,

$$\text{Reg}(\neg \mathcal{N}_t^s) \leq \sum_{t=1}^T 2t^{-2} \cdot \Delta_{\max} \leq \frac{\pi^2}{3} \cdot \Delta_{\max},$$

$$\text{Reg}(\mathcal{F}_t) \leq (1 - \beta)T \cdot \Delta_{\max}.$$

The filtered regret with null event

$$\begin{aligned}
 \text{Reg}(\{\}) &\leq \text{Reg}(\neg \mathcal{N}_t^s) + \text{Reg}(\mathcal{F}_t) + \text{Reg}(\Delta_{S_t} < M_{S_t}) + \text{Reg}(\{\Delta_{S_t} \geq M_{S_t}\} \wedge \neg \mathcal{F}_t \wedge \mathcal{N}_t^s) \\
 &\leq (1 - \beta)T \cdot \Delta_{\max} + \frac{\pi^2}{3} \cdot \Delta_{\max} + 2Bm + \sum_{i \in [m]} \frac{48B^2 K \ln T}{M_i} + \text{Reg}(\Delta_{S_t} < M_{S_t}).
 \end{aligned}$$

By definition of filtered regret,  $\text{Reg}_{\mu, \alpha, \beta}(T) = \text{Reg}(T, \{\}) - (1 - \beta)T \cdot \Delta_{\max}$ , so

$$\text{Reg}_{\mu, \alpha, \beta}(T) \leq \frac{\pi^2}{3} \cdot \Delta_{\max} + 2Bm + \sum_{i \in [m]} \frac{48B^2 K \ln T}{M_i} + \text{Reg}(\Delta_{S_t} < M_{S_t}).$$

For distribution-dependent bound, take  $M_i = \Delta_{\max}^i$ , then  $\text{Reg}(\Delta_{S_t} < M_{S_t}) = 0$  and we have

$$\text{Reg}_{\mu, \alpha, \beta}(T) \leq \sum_{i \in [m]} \frac{48B^2 K \ln T}{M_i} + 2Bm + \frac{\pi^2}{3} \cdot \Delta_{\max}.$$

For distribution-independent bound, take  $M_i = M = \sqrt{(48B^2mK \ln T)/T}$ , then  $\text{Reg}(\Delta_{S_t} < M_{S_t}) \leq TM$  and we have

$$\begin{aligned}
 \text{Reg}_{\mu, \alpha, \beta}(T) &\leq \sum_{i \in [m]} \frac{48B^2K \ln T}{M_i} + 2Bm + \frac{\pi^2}{3} \cdot \Delta_{\max} + \text{Reg}(\Delta_{S_t} < M_{S_t}) \\
 &\leq \frac{48B^2mK \ln T}{M} + 2Bm + \frac{\pi^2}{3} \cdot \Delta_{\max} + TM \\
 &= 2\sqrt{48B^2mKT \ln T} + \frac{\pi^2}{3} \cdot \Delta_{\max} + 2Bm \\
 &\leq 14B\sqrt{mKT \ln T} + \frac{\pi^2}{3} \cdot \Delta_{\max} + 2Bm.
 \end{aligned}$$

□

### C.3. Proof of Theorem 1 (1-Norm Case Regret Bound)

Let  $\mathcal{F}_t$  be the event  $\{r_{S_t}(\bar{\mu}) < \alpha \cdot \text{opt}(\bar{\mu})\}$ . In other words,  $\mathcal{F}_t$  means the oracle fails in round  $t$ . By assumption,  $\Pr\{\mathcal{F}_t\} \leq 1 - \beta$ .

To unify the proofs for distribution-dependent and distribution-independent bounds, we introduce a positive real number  $M_i$  for each arm  $i$ . Define  $M_S = \max_{i \in \tilde{S}} M_i$  for each action  $S$ , specifically,  $M_S = 0$  if  $\tilde{S} = \emptyset$ . To prove the distribution-dependent bound, we will let  $M_i = \Delta_{\min}^i$ . To prove the distribution-independent bound, we will let  $M_i = M = \tilde{\Theta}(T^{-1/2})$  to balance bounds for  $\text{Reg}(\{\Delta_{S_t} \geq M_{S_t}\})$  and  $\text{Reg}(\{\Delta_{S_t} < M_{S_t}\})$ . Implement definition of  $\mathcal{N}_t^i$  (Definition 7) with  $j_{\max}^i = j_{\max}(M_i) = \left\lceil \log_2 \frac{2BK}{M_i} \right\rceil_0$ . Define

$$\kappa_{j,T}(M, s) = \begin{cases} 2B \min\{1, 2 \cdot 2^{-j}\}, & \text{if } s = 0, \\ 2B \sqrt{\frac{\lambda_j \ln T}{s}}, & \text{if } 1 \leq s \leq \ell_{j,T}(M), \\ 0, & \text{if } s \geq \ell_{j,T}(M) + 1, \end{cases}$$

where

$$\lambda_j = \begin{cases} 6, & \text{if } j = 0, \\ 72 \cdot 2^{-j}, & \text{if } j > 0, \end{cases}$$

and

$$\ell_{j,T}(M) = \left\lfloor \frac{4\lambda_j B^2 K^2 \ln T}{M^2} \right\rfloor.$$

and the following lemma explains that  $\kappa$  is the contribution to regret.

**Lemma 7.** *In every run of the CUCB algorithm on a problem instance that satisfies 1-norm TPM bounded smoothness (Condition 2), for any vector  $\{M_i\}_{i \in [m]}$  of positive real numbers and  $1 \leq t \leq T$ , if  $\{\Delta_{S_t} \geq M_{S_t}\}, \neg \mathcal{F}_t, \mathcal{N}_t^s$  and  $\mathcal{N}_t^i$  hold, we have*

$$\Delta_{S_t} \leq \sum_{i \in \tilde{S}_t} \kappa_{j_i, T}(M_i, N_{i, j_i, t-1}),$$

where  $j_i$  is the index of the TP group with  $S_t \in \mathcal{S}_{i, j_i}$  (See Definition 5).

*Proof.* The right hand side of the inequality is non-negative, so it holds naturally if  $\Delta_{S_t} = 0$ . We only need to consider  $\Delta_{S_t} > 0$ . By  $\mathcal{N}_t^s$  and  $\neg \mathcal{F}_t$ , we have

$$r_{S_t}(\bar{\mu}_t) \geq \alpha \cdot \text{opt}(\bar{\mu}_t) \geq \alpha \cdot \text{opt}(\mu) = r_{S_t}(\mu) + \Delta_{S_t},$$

Then by Condition 2,

$$\Delta_{S_t} \leq r_{S_t}(\bar{\mu}_t) - r_{S_t}(\mu) \leq B \sum_{i \in \tilde{S}_t} p_i^{\mu, S_t} (\bar{\mu}_{i,t} - \mu_i).$$

We are going to bound  $\Delta_{S_t}$  by bounding  $p_i^{\mu, S_t}(\bar{\mu}_{i,t} - \mu_i)$ . But before doing so, we first perform a transformation. As we have  $\Delta_{S_t} \geq M_{S_t}$ , so  $B \sum_{i \in \tilde{S}_t} p_i^{\mu, S_t}(\bar{\mu}_{i,t} - \mu_i) \geq \Delta_{S_t} \geq M_{S_t}$ . We have

$$\begin{aligned}
 \Delta_{S_t} &\leq B \sum_{i \in \tilde{S}_t} p_i^{\mu, S_t}(\bar{\mu}_{i,t} - \mu_i) \\
 &\leq -M_{S_t} + 2B \sum_{i \in \tilde{S}_t} p_i^{\mu, S_t}(\bar{\mu}_{i,t} - \mu_i) \\
 &= 2B \sum_{i \in \tilde{S}_t} \left[ p_i^{\mu, S_t}(\bar{\mu}_{i,t} - \mu_i) - \frac{M_{S_t}}{2B|\tilde{S}_t|} \right] \\
 &\leq 2B \sum_{i \in \tilde{S}_t} \left[ p_i^{\mu, S_t}(\bar{\mu}_{i,t} - \mu_i) - \frac{M_{S_t}}{2BK} \right] \\
 &\leq 2B \sum_{i \in \tilde{S}_t} \left[ p_i^{\mu, S_t}(\bar{\mu}_{i,t} - \mu_i) - \frac{M_i}{2BK} \right]. \tag{12}
 \end{aligned}$$

We then bound  $p_i^{\mu, S_t}(\bar{\mu}_{i,t} - \mu_i)$ . By  $\mathcal{N}_t^s$ ,

$$\bar{\mu}_{i,t} - \mu_i \leq 2\rho_{i,t} = 2\sqrt{\frac{3 \ln t}{2T_{i,t-1}}}.$$

Both  $\bar{\mu}_{i,t}$  and  $\mu_i$  are in  $[0, 1]$ , so  $\bar{\mu}_{i,t} - \mu_i \leq 1$ . We then bound  $p_i^{\mu, S_t}(\bar{\mu}_{i,t} - \mu_i)$  in different cases.

- *Case I:*  $j_i = 0$ . There must be  $p_i^{\mu, S_t} = 1$  and  $T_{i,t-1} \geq N_{i,j_i,t-1}$ , as every time  $N_{i,j_i}$  is increased,  $T_i$  is also increased. So

$$\begin{aligned}
 p_i^{\mu, S_t}(\bar{\mu}_{i,t} - \mu_i) &= \bar{\mu}_{i,t} - \mu_i \leq \min \left\{ 2\sqrt{\frac{3 \ln t}{2T_{i,t-1}}}, 1 \right\} \\
 &\leq \min \left\{ \sqrt{\frac{6 \ln T}{N_{i,j_i,t-1}}}, 1 \right\}.
 \end{aligned}$$

- *Case II:*  $1 \leq j_i \leq j_{\max}^i$ . Then we have  $p_i^{\mu, S_t} < 2 \cdot 2^{-j_i}$ . If  $\sqrt{\frac{6 \ln t}{\frac{1}{3}N_{i,j_i,t-1} \cdot 2^{-j_i}}} \leq 1$ , by  $\mathcal{N}_t^1$ ,

$$\bar{\mu}_{i,t} - \mu_i \leq 2\sqrt{\frac{3 \ln t}{2T_{i,t-1}}} \leq \sqrt{\frac{6 \ln t}{\frac{1}{3}N_{i,j_i,t-1} \cdot 2^{-j_i}}},$$

so

$$\bar{\mu}_{i,t} - \mu_i \leq \min \left\{ \sqrt{\frac{6 \ln t}{\frac{1}{3}N_{i,j_i,t-1} \cdot 2^{-j_i}}}, 1 \right\},$$

and

$$\begin{aligned}
 &p_i^{\mu, S_t}(\bar{\mu}_{i,t} - \mu_i) \\
 &\leq 2 \cdot 2^{-j_i} \cdot \min \left\{ \sqrt{\frac{6 \ln t}{\frac{1}{3}N_{i,j_i,t-1} \cdot 2^{-j_i}}}, 1 \right\} \\
 &= \min \left\{ \sqrt{\frac{72 \cdot 2^{-j_i} \ln T}{N_{i,j_i,t-1}}}, 2 \cdot 2^{-j_i} \right\}.
 \end{aligned}$$



In conclusion of the above two cases, if  $j_i \leq j_{\max}^i$ , then

$$p_i^{\mu, S_t}(\bar{\mu}_{i,t} - \mu_i) \leq \min \left\{ \sqrt{\frac{\lambda_{j_i} \ln T}{N_{i,j_i,t-1}}}, 2 \cdot 2^{-j_i}, 1 \right\}.$$

If  $N_{i,j_i,t-1} \geq \ell_{j_i,T}(M_i) + 1$ , then  $\sqrt{\frac{72 \cdot 2^{-j_i} \ln T}{N_{i,j_i,t-1}}} \leq \frac{M_i}{2BK}$  and  $p_i^{\mu, S_t}(\bar{\mu}_{i,t} - \mu_i) - \frac{M_i}{2BK} \leq 0$ . If  $N_{i,j_i,t-1} = 0$ , we use the bound  $p_i^{\mu, S_t}(\bar{\mu}_{i,t} - \mu_i) \leq \min\{2 \cdot 2^{-j_i}, 1\}$ . Otherwise, i.e.  $1 \leq N_{i,j_i,t-1} \leq \ell_{j_i,T}(M_i)$ , we use  $p_i^{\mu, S_t}(\bar{\mu}_{i,t} - \mu_i) \leq \sqrt{\frac{\lambda_{j_i} \ln T}{N_{i,j_i,t-1}}}$ . Recall the definition of  $\kappa_{j,T}(M, s)$ , then, for  $j_i \leq j_{\max}^i$ , we have

$$p_i^{\mu, S_t}(\bar{\mu}_{i,t} - \mu_i) - \frac{M_i}{2BK} \leq \frac{1}{2B} \kappa_{j_i,T}(M_i, N_{i,j_i,t-1}). \quad (13)$$

- *Case III:*  $j_i \geq j_{\max}^i + 1 = \left\lceil \log_2 \frac{2BK}{M_i} \right\rceil + 1$ . Then we have

$$\begin{aligned} p_i^{\mu, S_t}(\bar{\mu}_{i,t} - \mu_i) &\leq p_i^{\mu, S_t} < 2 \cdot 2^{-j_i} \\ &\leq 2 \cdot 2^{-\log_2 \frac{2BK}{M_i} - 1} = \frac{M_i}{2BK}. \end{aligned}$$

So

$$p_i^{\mu, S_t}(\bar{\mu}_{i,t} - \mu_i) - \frac{M_i}{2BK} < 0 \leq \frac{1}{2B} \kappa_{j_i,T}(M_i, N_{i,j_i,t-1}). \quad (14)$$

Combining Eq. (12), (13) and (14), we conclude the proof with

$$\begin{aligned} \Delta_{S_t} &\leq 2B \sum_{i \in \tilde{S}_t} \left[ p_i^{\mu, S_t}(\bar{\mu}_{i,t} - \mu_i) - \frac{M_i}{2BK} \right] \\ &\leq \sum_{i \in \tilde{S}_t} \kappa_{j_i,T}(M_i, N_{i,j_i,t-1}). \end{aligned} \quad \square$$

We remark that the proof of Lemma 7, in particular the derivation leading to Eq. (12) together with the argument in the paragraph before Eq.(13), contains the reverse amortization trick we mentioned in the main text. In particular, by the derivation of Eq. (12), the contribution of every arm  $i$  to regret  $\Delta_{S_t}$  is accounted as  $2B \left[ p_i^{\mu, S_t}(\bar{\mu}_{i,t} - \mu_i) - \frac{M_i}{2BK} \right]$ . Then by the argument in the paragraph before Eq.(13), if  $N_{i,j_i,t-1} \geq \ell_{j_i,T}(M_i) + 1$ , meaning that  $i$  has been triggered by actions in group  $j_i$  for at least  $\ell_{j_i,T}(M_i) + 1$ , its error  $|\bar{\mu}_{i,t} - \mu_i|$  would be small enough such that its contribution to the regret  $\Delta_{S_t}$  is not positive. This trick eliminates the need of summing up small errors from many sufficiently sampled arms, leading to a tighter regret bound. The same trick can be seen in Appendix C.2, Eq.(11) and the derivation that follows for the no triggered arm case.

**Lemma 8.** *For the CUCB algorithm on a problem instance that satisfies TPM bounded smoothness with 1-norm (Condition 2),*

$$\text{Reg}(\{\Delta_{S_t} \geq M_{S_t}\} \wedge \neg \mathcal{F}_t \wedge \mathcal{N}_t^s \wedge \mathcal{N}_t^l) \leq \sum_{i \in [m]} \frac{624a^2 K \ln T}{M_i} + 6am.$$

*Proof.* We bound  $\text{Reg}(\{\Delta_{S_t} \geq M_{S_t}\} \wedge \neg \mathcal{F}_t \wedge \mathcal{N}_t^s \wedge \mathcal{N}_t^l)$  with Lemma 7. In every run,

$$\begin{aligned} \sum_{t=1}^T \mathbb{I}(\{\Delta_{S_t} \geq M_{S_t}\} \wedge \neg \mathcal{F}_t \wedge \mathcal{N}_t^s \wedge \mathcal{N}_t^l) \Delta_{S_t} &\leq \sum_{t=1}^T \sum_{i \in \tilde{S}_t} \kappa_{j_i,T}(M_i, N_{i,j_i,t-1}) \\ &= \sum_{i \in [m]} \sum_{j=0}^{+\infty} \sum_{s=0}^{N_{i,j,T}-1} \kappa_{j,T}(M_i, s), \end{aligned} \quad (15)$$

where (15) is due to  $N_{i,j_i}$  is increased if and only if  $i \in \tilde{S}_t$ . For every arm  $i$  and  $j \geq 0$ ,

$$\sum_{s=0}^{N_{i,j,T}-1} \kappa_{j,T}(M_i, s) \leq \sum_{s=0}^{\ell_{j,T}(M_i)} \kappa_{j,T}(M_i, s) \quad (16)$$

$$\begin{aligned} &= \kappa_{j,T}(M_i, 0) + \sum_{s=1}^{\ell_{j,T}(M_i)} \kappa_{j,T}(M_i, s) \\ &= \kappa_{j,T}(M_i, 0) + \sum_{s=1}^{\ell_{j,T}(M_i)} 2a \sqrt{\frac{\lambda_j \ln T}{s}} \\ &\leq \kappa_{j,T}(M_i, 0) + 4a \sqrt{\lambda_j \ln T} \sqrt{\ell_{j,T}(M_i)}, \end{aligned} \quad (17)$$

where (16) is due to  $\kappa_{j,T}(s) = 0$  when  $s \geq \ell_{j,T}(M) + 1$ , and (17) is due to the fact that, for every natural number integer  $n$ ,

$$\sum_{s=1}^n \sqrt{\frac{1}{s}} \leq \int_{s=0}^n \sqrt{\frac{1}{s}} ds = 2\sqrt{n}.$$

By definition,  $\ell_{j,T}(M_i) \leq \frac{4\lambda_j a^2 K^2 \ln T}{M_i^2}$ , so

$$\begin{aligned} (17) &\leq \kappa_{j,T}(M, 0) + 4a \sqrt{\lambda_j \ln T} \sqrt{\frac{4\lambda_j a^2 K^2 \ln T}{M_i^2}} \\ &= 2a \min\{1, 2 \cdot 2^{-j}\} + \frac{8a^2 K \lambda_j \ln T}{M_i}. \end{aligned}$$

Then we continue (15) with

$$\begin{aligned} (15) &\leq \sum_{i \in [m]} \sum_{j=0}^{+\infty} \left( 2a \min\{1, 2 \cdot 2^{-j}\} + \frac{4a^2 K \lambda_j \ln T}{M_i} \right) \\ &= \sum_{i \in [m]} \left[ 2a \cdot \left( 1 + \sum_{j=1}^{+\infty} 2 \cdot 2^{-j} \right) + \frac{8a^2 K \ln T}{M_i} \cdot \left( 6 + \sum_{j=1}^{+\infty} 72 \cdot 2^{-j} \right) \right] \\ &= \sum_{i \in [m]} \left[ 6a + 72 \cdot \frac{8a^2 K \ln T}{M_i} \right] \\ &= \sum_{i \in [m]} \frac{624a^2 K \ln T}{M_i} + 6am. \end{aligned}$$

By taking expectation over all possible runs,

$$\text{Reg}(\{\Delta_{S_t} \geq M_{S_t}\} \wedge \neg \mathcal{F}_t \wedge \mathcal{N}_t^s \wedge \mathcal{N}_t^t) = \mathbb{E}[\mathbb{I}(\{\Delta_{S_t} \geq M\} \wedge \neg \mathcal{F}_t \wedge \mathcal{N}_t^s \wedge \mathcal{N}_t^t) \Delta_{S_t}] \leq \sum_{i \in [m]} \frac{624a^2 K \ln T}{M_i} + 6am.$$

□

*Proof of Theorem 1.* Recall Definition 3, the definition of event-filtered regret:

$$\text{Reg}_{\mu}^A(T, \{\mathcal{E}_t\}_{t \geq 1}) = \mathbb{E} \left[ \sum_{t=1}^T \mathbb{I}(\mathcal{E}_t) (\alpha \cdot \text{opt}_{\mu} - r_{S_t^A}(\mu)) \right] = T \cdot \alpha \cdot \text{opt}_{\mu} - \mathbb{E} \left[ \sum_{t=1}^T \mathbb{I}(\mathcal{E}_t) (r_{S_t^A}(\mu)) \right].$$

Then for filtered regret with null event (the event that is always true), we have  $\text{Reg}(\{\}) = \text{Reg}_{\mu, \alpha, \beta} + (1 - \beta)T \cdot \alpha \cdot \text{opt}_{\mu}$ . We divide this filtered regret into parts as

$$\text{Reg}(\{\}) \leq \text{Reg}(\{\Delta_{S_t} < M_{S_t}\}) + \text{Reg}(\mathcal{F}_t) + \text{Reg}(\neg \mathcal{N}_t^s) + \text{Reg}(\neg \mathcal{N}_t^t) + \text{Reg}(\{\Delta_{S_t} \geq M_{S_t}\} \wedge \neg \mathcal{F}_t \wedge \mathcal{N}_t^s \wedge \mathcal{N}_t^t). \quad (18)$$

By definition of filtered regret,  $Reg(\mathcal{E}_t) \leq \sum_{t=1}^T \mathbb{I}\{\mathcal{E}_t\} \Delta_{S_t} \leq \sum_{t=1}^T \Pr\{\mathcal{E}_t\} \cdot \Delta_{\max}$ , then

$$Reg(\mathcal{F}_t) \leq \sum_{t=1}^T \Pr\{\mathcal{F}_t\} \Delta_{\max} = (1 - \beta)T \cdot \Delta_{\max}, \quad (19)$$

$$Reg(\neg \mathcal{N}_t^s) \leq \sum_{t=1}^T \Pr\{\neg \mathcal{N}_t^s\} \Delta_{\max} \leq \frac{\pi^2}{3} \cdot m \cdot \Delta_{\max}, \quad (20)$$

$$Reg(\neg \mathcal{N}_t^l) \leq \sum_{t=1}^T \Pr\{\neg \mathcal{N}_t^l\} \Delta_{\max} \leq \frac{\pi^2}{6} \cdot \sum_{i \in [m]} j_{\max}^i \cdot \Delta_{\max}. \quad (21)$$

By Lemma 8,

$$Reg(\{\Delta_{S_t} \geq M_{S_t}\} \wedge \neg \mathcal{F}_t \wedge \mathcal{N}_t^s \wedge \mathcal{N}_t^l) \leq \sum_{i \in [m]} \frac{624a^2 K \ln T}{M_i} + 6am.$$

Take  $M_i = \Delta_{\min}^i$ . If  $\Delta_{S_t} < M_{S_t}$ , then  $\Delta_{S_t} = 0$ , since we have either  $\tilde{S}_t = \emptyset$  or  $\Delta_{S_t} < M_{S_t} \leq M_i$  for some  $i \in \tilde{S}_t$ . So  $Reg(\{\Delta_{S_t} < M_{S_t}\}) = 0$ . Then we have

$$Reg(\{\}) \leq (1 - \beta)T \cdot \Delta_{\max} + \sum_{i \in [m]} \frac{624a^2 K \ln T}{\Delta_{\min}^i} + 6am + \frac{\pi^2}{6} \cdot \sum_{i \in [m]} (j_{\max}(\Delta_{\min}^i) + 2) \cdot \Delta_{\max}, \quad (22)$$

where we abuse the notation of  $j_{\max}(M) = \left\lceil \log_2 \frac{2BK}{M_i} \right\rceil_0$ .

On the other hand, take  $M_i = M = \sqrt{(624a^2 m K \ln T)/T}$ , then  $\Delta_{S_t}$  is also  $M$  for every action  $S_t$  that  $\tilde{S}_t$  is non-empty. We bound  $Reg(\{\Delta_{S_t} < M\})$  with

$$Reg(\{\Delta_{S_t} < M_{S_t}\}) = \sum_{t=1}^T \mathbb{I}\{\Delta_{S_t} < M_{S_t}\} \Delta_{S_t} \leq \sum_{t=1}^T \mathbb{I}\{\Delta_{S_t} < M_{S_t}\} M \leq TM.$$

So the filtered regret with null event is bounded by

$$\begin{aligned} Reg(\{\}) &\leq (1 - \beta)T \cdot \Delta_{\max} + \frac{624a^2 m K \ln T}{M} + 6am + TM + \frac{\pi^2}{6} \cdot (j_{\max}(M) + 2) \cdot m \cdot \Delta_{\max} \\ &= (1 - \beta)T \cdot \Delta_{\max} + \frac{624a^2 m K \ln T}{\sqrt{(624a^2 m K \ln T)/T}} + 6am + T \sqrt{(624a^2 m K \ln T)/T} + \frac{\pi^2}{6} \cdot (j_{\max}(M) + 2) \cdot m \cdot \Delta_{\max} \\ &\leq (1 - \beta)T \cdot \Delta_{\max} + 50a \sqrt{m K T \ln T} + 6am + \frac{\pi^2}{6} \cdot (j_{\max}(M) + 2) \cdot m \cdot \Delta_{\max}. \end{aligned} \quad (23)$$

Since  $Reg_{\mu, \alpha, \beta} = Reg(\{\}) - (1 - \beta)T \cdot \alpha \cdot \text{opt}_{\mu} \leq Reg(\{\}) - (1 - \beta)T \cdot \Delta_{\max}$ , (22) implies (1) and (23) implies (2).  $\square$

#### C.4. Proof of Theorem 2

**Lemma 9.** *In every run of the CUCB algorithm on a problem instance that satisfies 1-norm RTPM bounded smoothness (Condition 3), for any vector  $\{M_i\}_{i \in [m]}$  of positive real numbers and  $1 \leq t \leq T$ , if  $\{\Delta_{S_t} \geq M_{S_t}\}$ ,  $\{\max_{i \in [m]} \rho_{i,t} \leq \frac{1}{4m}\}$ ,  $\neg \mathcal{F}_t$ ,  $\mathcal{N}_t^s$  and  $\mathcal{N}_t^l$  hold, we have*

$$\Delta_{S_t} \leq \sum_{i \in \tilde{S}_t} \kappa_{j_i, T}(M_i, N_{i, j_i, t-1}),$$

where  $j_i$  is the index of the TP group with  $S_t \in \mathcal{S}_{i, j_i}$  (See Definition 5).

*Proof.* The proof is very similar to the proof of Lemma 7. As we have  $\{\max_{i \in [m]} \rho_{i,t} \leq \frac{1}{4m}\}$  and  $\mathcal{N}_t^s$ , for every arm  $i$ ,  $|\bar{\mu}_{i,t-1} - \mu_i| \leq 2\rho_{i,t} \leq \frac{1}{2m}$ . Then the additional restriction in RTPM bounded smoothness is satisfied. So we can replace the usage of TPM bounded smoothness (Condition 2) in the proof to Lemma 7 with RTPM bounded smoothness. The remaining part of the proof is the same.  $\square$

**Lemma 10.** *For the CUCB algorithm on a problem instance that satisfies RTPM bounded smoothness with 1-norm (Condition 3),*

$$\text{Reg}\left(\{\Delta_{S_t} \geq M_{S_t}\} \wedge \left\{\max_{i \in [m]} \rho_{i,t} \leq \frac{1}{4m}\right\} \wedge \neg \mathcal{F}_t \wedge \mathcal{N}_t^s \wedge \mathcal{N}_t^r\right) \leq \sum_{i \in [m]} \frac{624a^2 K \ln T}{M_i} + 6am.$$

*Proof.* This lemma is RTPM bounded smoothness version of Lemma 8. With  $\{\max_{i \in [m]} \rho_{i,t} \leq \frac{1}{4m}\}$ , the remaining part of proof is the same.  $\square$

*Proof of Theorem 2.*

$$\text{reg}(\{\}) = \text{Reg}(\{\Delta_{S_t} < M_{S_t}\}) + \text{Reg}(\mathcal{F}_t) + \text{Reg}(\neg \mathcal{N}_t^s) + \text{Reg}(\neg \mathcal{N}_t^r) + \text{Reg}(\{\max_{i \in [m]} \rho_{i,t} > \frac{1}{4m}\}) + \text{Reg}(\{\Delta_{S_t} \geq M\} \wedge \neg \mathcal{F}_t \wedge \mathcal{N}_t^s \wedge \mathcal{N}_t^r).$$

By (19), (20) and (21),

$$\text{Reg}(\{\Delta_{S_t} < M_{S_t}\}) + \text{Reg}(\mathcal{F}_t) + \text{Reg}(\neg \mathcal{N}_t^s) \leq (1 - \beta)T \cdot \Delta_{\max} + \frac{\pi^2}{6} \cdot \sum_{i \in [m]} (j_{\max}(\Delta_{\min}^i) + 2) \cdot \Delta_{\max}.$$

By Lemma 10,

$$\text{Reg}\left(\{\Delta_{S_t} \geq M_{S_t}\} \wedge \left\{\max_{i \in [m]} \rho_{i,t} \leq \frac{1}{4m}\right\} \wedge \neg \mathcal{F}_t \wedge \mathcal{N}_t^s \wedge \mathcal{N}_t^r\right) \leq \sum_{i \in [m]} \frac{624a^2 K \ln T}{M_i} + 6am.$$

By Lemma 6,

$$\text{Reg}(\{\max_{i \in [m]} \rho_{i,t} \geq \frac{1}{4m}\}) \leq (24m^3 \ln T + m) \cdot \Delta_{\max}.$$

With the same method as in the proof of Theorem 2, we take  $M_i = \Delta_{\min}^i$  and have (3). And we take  $M_i = M = \sqrt{(624a^2 m K \ln T)/T}$  and have (4).  $\square$

## D. Results with $\infty$ -norm TPM Conditions

### D.1. TPM Conditions with the $\infty$ -norm

We first restate the original bounded smoothness condition in (Chen et al., 2016b) below, which is an  $\infty$ -norm based condition.

**Condition 7 (Bounded Smoothness).** *We say that a CMAB-T problem instance satisfies bounded smoothness, if there exists a continuous, strictly increasing (and thus invertible) function  $f(\cdot)$  with  $f(0) = 0$ , such that for any two distributions  $D, D' \in \mathcal{D}$  with expectation vectors  $\boldsymbol{\mu} = (\mu_1, \dots, \mu_m)$  and  $\boldsymbol{\mu}' = (\mu'_1, \dots, \mu'_m)$ , and for any  $\Lambda > 0$ , we have  $|r_{\boldsymbol{\mu}}(S) - r_{\boldsymbol{\mu}'}(S)| \leq f(\Lambda)$  if  $\max_{i \in \tilde{S}} |\mu_i - \mu'_i| \leq \Lambda$ , for all  $S \in \mathcal{S}$ , where  $\tilde{S} = \{i \in [m] \mid \Pr_{X \sim D, \tau}\{i \in \tau(S, X)\} > 0\}$  is the set of arms that could be triggered by action  $S$ .*

Note that  $f(\cdot)$  may depend on problem instance parameters such as  $m$ , but not on action  $S$  or mean vectors  $\boldsymbol{\mu}, \boldsymbol{\mu}'$ .

Similar to the 1-norm case, we use triggering probabilities to moderate the bounded smoothness condition to obtain the following TPM version:

**Condition 8. ( $\infty$ -Norm TPM Bounded Smoothness)** *We say a CMAB-T problem instance satisfies the triggering-probability-moderated (TPM) bounded smoothness with bounded smoothness function  $f(x)$ , if for any two distributions  $D, D' \in \mathcal{D}$  with expectation vectors  $\boldsymbol{\mu}$  and  $\boldsymbol{\mu}'$ , any action  $S$  and any  $\Lambda > 0$ , we have  $|r_S(\boldsymbol{\mu}) - r_S(\boldsymbol{\mu}')| \leq f(\Lambda)$  if  $\max_{i \in [m]} p_i^{\boldsymbol{\mu}, S} |\mu_i - \mu'_i| \leq \Lambda$ .*

Note that Condition 8 is stronger than Condition 7 under the same bounded smoothness function  $f$ . This is because if we have  $\max_{i \in [m]} |\mu_i - \mu'_i| \leq \Lambda$ , then we have  $\max_{i \in [m]} p_i^{\boldsymbol{\mu}, S} |\mu_i - \mu'_i| \leq \Lambda$ . Then if Condition 8 holds, we have  $|r_S(\boldsymbol{\mu}) - r_S(\boldsymbol{\mu}')| \leq f(\Lambda)$ . This means that if Condition 8 holds, we have  $|r_S(\boldsymbol{\mu}) - r_S(\boldsymbol{\mu}')| \leq f(\Lambda)$  if  $\max_{i \in [m]} |\mu_i - \mu'_i| \leq \Lambda$ , which is exactly Condition 7.



Comparing Condition 8 with Condition 4, we see that in Condition 4,  $\mu$  and  $\mu'$  only differ in one dimension  $i$ , while in Condition 8  $\mu$  and  $\mu'$  differ in all dimensions. This is the reason we refer to Condition 4 as local TPM bounded smoothness.

Similar to the 1-norm RTPM bounded smoothness (Condition 3), we could also have the following slightly relaxed version for the  $\infty$ -norm:

**Condition 9. ( $\infty$ -Norm RTPM Bounded Smoothness)** We say that a CMAB-T problem instance satisfies restricted triggering-probability-moderated (RTPM) bounded smoothness with bounded smoothness function  $f(x)$ , if for any two distributions  $D, D' \in \mathcal{D}$  with expectation vectors  $\mu$  and  $\mu'$ , any action  $S$ , and any  $\Lambda > 0$ , we have  $|r_S(\mu) - r_S(\mu')| \leq f(\Lambda)$  if  $\max_i p_i^{\mu, S} |\mu_i - \mu'_i| \leq \Lambda$  and  $\max_i |\mu_i - \mu'_i| \leq \frac{1}{2m}$ .

Again similar to the 1-norm case, we could use the two local conditions (Conditions 4 and 5) to achieve the  $\infty$ -norm RTPM bounded smoothness, as stated in the following lemma:

**Lemma 11.** If a CMAB-T problem instance satisfies both the local TPM bounded smoothness (Condition 4) with a local bounded smoothness function  $g(x)$  and the local moderated continuity on triggering probability (Condition 5), then it satisfies the  $\infty$ -norm RTPM bounded smoothness (Condition 9) with bounded smoothness function  $f(x) = m \cdot g(2x)$ .

The proof of the above lemma essentially is the same as the proof of Lemma 3, except that at Inequality (5), we continue with

$$\begin{aligned} |r_S(\mu) - r_S(\mu')| &\leq \sum_{j=1}^m g(2p_j^{\mu, S} |\mu_j - \mu'_j|) \\ &\leq \sum_{j=1}^m g(2\Lambda) \\ &= m \cdot g(2\Lambda). \end{aligned}$$

## D.2. Theorem and Proofs with $\infty$ -norm TPM Conditions

**Theorem 6.** Suppose a CMAB-T problem instance  $([m], \mathcal{S}, \mathcal{D}, D^{\text{trig}}, R)$  satisfies monotonicity (Condition 1). For a fixed environment instance  $D \in \mathcal{D}$  with expectation vector  $\mu$ , the  $T$ -round  $(\alpha, \beta)$ -approximation regret bound using an  $(\alpha, \beta)$ -approximation oracle in various cases are given below.

- (1) For the CUCB algorithm on a problem instance that satisfies TPM bounded smoothness (Condition 8) with bounded smoothness function  $f(x)$ , together with  $\Delta_{\min} > 0$ , the regret is at most

$$\begin{aligned} &\sum_{i \in [m]} 78 \ln T \left( \frac{\Delta_{\min}^i}{f^{-1}(\Delta_{\min}^i)^2} + \int_{\Delta_{\min}^i}^{\Delta_{\max}^i} \frac{1}{f^{-1}(x)^2} dx \right) \\ &+ m \cdot \left[ \left( \frac{\pi^2}{6} + 1 \right) \lceil -\log_2 f^{-1}(\Delta_{\min}) \rceil_0 + \frac{\pi^2}{3} + 1 \right] \cdot \Delta_{\max}; \end{aligned}$$

- (2) For the CUCB algorithm on a problem instance that satisfies TPM bounded smoothness (Condition 8) with bounded smoothness function  $f(x) = ax$ , the regret is at most

$$25a\sqrt{mT \ln T} + m \cdot \left[ \left( \frac{\pi^2}{6} + 1 \right) \lceil -\log_2(\sqrt{156m \ln T/T}) \rceil_0 + \frac{\pi^2}{3} + 1 \right] \cdot \Delta_{\max};$$

- (3) For the CUCB-IS algorithm on a problem instance that satisfies the RTPM bounded smoothness (Condition 9) with bounded smoothness function  $f(x)$ , together with  $\Delta_{\min} > 0$ , the regret is at most

$$\begin{aligned} &\sum_{i \in [m]} 78 \ln T \left( \frac{\Delta_{\min}^i}{f^{-1}(\Delta_{\min}^i)^2} + \int_{\Delta_{\min}^i}^{\Delta_{\max}^i} \frac{1}{f^{-1}(x)^2} dx \right) \\ &+ m \cdot \left[ 24m^2 \ln T + \left( \frac{\pi^2}{6} + 1 \right) \lceil -\log_2 f^{-1}(\Delta_{\min}) \rceil_0 + \frac{\pi^2}{3} + 2 \right] \cdot \Delta_{\max}; \end{aligned}$$

(4) For the CUCB-IS algorithm on a problem instance that satisfies RTPM global bounded smoothness (Condition 9) with bounded smoothness function  $f(x) = ax$ , the regret is at most

$$25a\sqrt{mT \ln T} + m \cdot \left[ 24m^2 \ln T + \left( \frac{\pi^2}{6} + 1 \right) \left[ -\log_2(\sqrt{156m \ln T/T}) \right]_0 + \frac{\pi^2}{3} + 2 \right] \cdot \Delta_{\max}.$$

We have several remarks on Theorem 6. First, the condition  $\Delta_{\min} > 0$  automatically holds if the action space  $\mathcal{S}$  is finite. Thus it is not an extra condition comparing to the result in (Chen et al., 2016b) when actions are set of base arms. If  $\Delta_{\min}$  is zero due to infinite  $\mathcal{S}$ , then we do not have regret bounds as in (1) and (3), but we still have regret bounds as in (2) and (4). Second, the regret bounds in (1) and (3) are distribution-dependent bounds, since it depends on  $\Delta_{\min}^i$ , which is determined by the distribution  $D$ ; regret bounds in (2) and (4) are distribution-independent bounds, since  $\Delta_{\max}$  can be easily replaced by a quantity only depending on the problem instance, such as the maximum possible reward value. Third, when  $\Delta_{\min}^i = +\infty$ ,  $\frac{\Delta_{\min}^i}{f^{-1}(\Delta_{\min}^i)^2} = 0$ .

#### D.2.1. PROOF OF THEOREM 6

In this subsection, we focus on giving a roadmap to prove Theorem 6 and showing the new techniques we invented to improve the regret bound. The remaining part of the proof is roughly the new calculation based on the old techniques (c.f. (Chen et al., 2016b)).

In this subsection, we omit  $(\alpha, \beta)$ -approximation for clarity, in other words, we assume  $\alpha = \beta = 1$ . Generalization to accommodate  $(\alpha, \beta)$  approximation can be found in the discussion section.

To exploit the advantage of TPM and RTPM bounded smoothness conditions (Conditions 8 & 9), for each arm  $i$ , we divide actions into groups according to  $p_i^{\mu, S}$ .

For convenience, we also allow to index the counters with  $q_i^{\mu, S_t} > 0$ , such that  $N_{i, q_i^{\mu, S_t}}$  indicates the same counter as  $N_{i, j}$  with  $q_i^{\mu, S_t} = 2^{-j}$ .

We use a shorthand as follows. For every arm  $i$  and action  $S$ , define

$$q_i^{\mu, S} = \begin{cases} 2^{-j}, & \text{if } S \in \mathcal{S}_{i, j}^{\mu}, \\ 0, & \text{if } p_i^{\mu, S} = 0. \end{cases}$$

**Definition 8.**

$$\ell_t(\Delta, q) = \begin{cases} 0, & \text{if } q \leq \frac{1}{2}f^{-1}(\Delta), \\ \lfloor \frac{6 \ln t}{f^{-1}(\Delta)^2} \rfloor + 1, & \text{if } q = 1, \\ \lfloor \frac{72q \ln t}{f^{-1}(\Delta)^2} \rfloor + 1, & \text{otherwise.} \end{cases}$$

To unify the proofs for distribution-dependent and distribution-independent bounds, we introduce a positive real number  $M$ . To prove the distribution-dependent bound, we will let  $M = \Delta_{\min}$  or  $M = \Delta_{\min}^i$  in some circumstances. To prove the distribution-independent bound, we will let  $M = \tilde{\Theta}(T^{-1/2})$  to balance bounds for  $\text{Reg}(\{\Delta_{S_t} \geq M\})$  and  $\text{Reg}(\{\Delta_{S_t} < M\})$ . And we implement  $\mathcal{N}_t^i$  (Definition 7) with  $j_{\max}^i = j_{\max}(M) = \lceil -\log_2 f^{-1}(M) \rceil_0$ . The following are three technical claims used in the main proof, and we define the proofs of these claims to Section D.2.2.

**Claim 1 (Bound of insufficiently sampled regret).** For any CMAB-T problem instance, any bounded smoothness function  $f(x)$ , any algorithm, any arm  $i$ , any natural number  $j$  and any positive real number  $M$ ,

$$\text{Reg}(\{\Delta_{S_t} \geq M, S_t \in \mathcal{S}_{i, j}, N_{i, j, t-1} < \ell_T(\Delta_{S_t}, 2^{-j})\}) \leq \ell_T(M, 2^{-j})M + \int_M^{\max\{\Delta_{\max}^i, M\}} \ell_T(x, 2^{-j}) dx.$$

**Claim 2 (Bound of sufficiently sampled regret for CUCB).** For the CUCB algorithm on a problem instance that satisfies TPM bounded smoothness (Condition 8) with bounded smoothness function  $f(x)$ ,

$$\text{Reg}(\{\Delta_{S_t} \geq M, \forall i, N_{i, q_i^{\mu, S_t}, t-1} \geq \ell_T(\Delta_{S_t}, q_i^{\mu, S_t})\}) \leq m \cdot (\lceil -\log_2 f^{-1}(M) \rceil_0 + 2) \cdot \frac{\pi^2}{6} \cdot \Delta_{\max}.$$

**Claim 3 (Bound of sufficiently sampled regret for CUCB-IS).** *For the CUCB-IS algorithm on a problem instance that satisfies the RTPM bounded smoothness (Condition 9) with bounded smoothness function  $f(x)$ ,*

$$\text{Reg}(\{\Delta_{S_t} \geq M, \forall i, \rho_{i,t} \leq \frac{1}{4m}, N_{i,q_i^{S_t},t-1} \geq \ell_T(\Delta_{S_t}, q_i^{S_t})\}) \leq m \cdot (\lceil -\log_2 f^{-1}(M) \rceil_0 + 2) \cdot \frac{\pi^2}{6} \cdot \Delta_{\max}.$$

We continue the proof of Theorem 6. Fix a value  $M > 0$ , we have

$$\begin{aligned} \text{Reg}(\{\}) &= \text{Reg}(\{\Delta_{S_t} < M\}) + \text{Reg}(\{\Delta_{S_t} \geq M\}) \\ &= \text{Reg}(\{\Delta_{S_t} < M\}) + \text{Reg}(\{\Delta_{S_t} \geq M, \forall i, N_{i,q_i^{S_t},t-1} \geq \ell_T(\Delta_{S_t}, q_i^{S_t})\}) \\ &\quad + \text{Reg}(\{\Delta_{S_t} \geq M, \exists i, N_{i,q_i^{S_t},t-1} < \ell_T(\Delta_{S_t}, q_i^{S_t})\}) \\ &\leq \text{Reg}(\{\Delta_{S_t} < M\}) + \text{Reg}(\{\Delta_{S_t} \geq M, \forall i, N_{i,q_i^{S_t},t-1} \geq \ell_T(\Delta_{S_t}, q_i^{S_t})\}) \\ &\quad + \sum_{i \in [m]} \text{Reg}(\{\Delta_{S_t} \geq M, N_{i,q_i^{S_t},t-1} < \ell_T(\Delta_{S_t}, q_i^{S_t})\}) \\ &\leq \text{Reg}(\{\Delta_{S_t} < M\}) + \text{Reg}(\{\Delta_{S_t} \geq M, \forall i, N_{i,q_i^{S_t},t-1} \geq \ell_T(\Delta_{S_t}, q_i^{S_t})\}) \\ &\quad + \sum_{i \in [m]} \sum_{j \geq 0} \text{Reg}(\{\Delta_{S_t} \geq M, S_t \in \mathcal{S}_{i,j}, N_{i,q_i^{S_t},t-1} < \ell_T(\Delta_{S_t}, q_i^{S_t})\}) \\ &= \text{Reg}(\{\Delta_{S_t} < M\}) + \text{Reg}(\{\Delta_{S_t} \geq M, \forall i, N_{i,q_i^{S_t},t-1} \geq \ell_T(\Delta_{S_t}, q_i^{S_t})\}) \\ &\quad + \sum_{i \in [m]} \sum_{j \geq 0} \text{Reg}(\{\Delta_{S_t} \geq M, S_t \in \mathcal{S}_{i,j}, N_{i,j,t-1} < \ell_T(\Delta_{S_t}, 2^{-j})\}). \end{aligned} \quad (24)$$

For the last part, if  $j \geq \lceil -\log_2 f^{-1}(M) \rceil_0 + 1$ , then  $2^{-j} \leq \frac{1}{2}f^{-1}(M)$  and

$$\frac{1}{2}f^{-1}(\Delta_{S_t}) \geq \frac{1}{2}f^{-1}(M) \geq 2^{-j}.$$

By Definition 8,  $\ell_T(\Delta_{S_t}, 2^{-j}) = 0$ . Then  $N_{i,j,t-1} < \ell_T(\Delta_{S_t}, 2^{-j})$  is impossible, so

$$\sum_{j \geq \lceil -\log_2 f^{-1}(M) \rceil_0 + 1} \text{Reg}(\{\Delta_{S_t} \geq M, S_t \in \mathcal{S}_{i,j}, N_{i,j,t-1} < \ell_T(\Delta_{S_t}, 2^{-j})\}) = 0.$$

**Lemma 12.** *For every arm  $i$ , the event-filtered regret*

$$\begin{aligned} &\sum_{j \geq 0} \text{Reg}(\{\Delta_{S_t} \geq M, S_t \in \mathcal{S}_{i,j}, N_{i,j,t-1} < \ell_T(\Delta_{S_t}, 2^{-j})\}) \\ &\leq 78 \ln T \left( \frac{M}{f^{-1}(M)^2} + \int_M^{\max\{\Delta_{\max}^i, M\}} \frac{1}{f^{-1}(x)^2} dx \right) + (j_{\max}(M) + 1) \cdot \Delta_{\max}^i. \end{aligned} \quad (25)$$

*Proof.* If  $M > \Delta_{\max}^i$ , it is impossible to have  $\Delta_{S_t} \geq M$  and  $S_t \in \mathcal{S}_{i,j}$  at the same time and then (25) = 0. Then the lemma holds trivially. So we may assume that  $M \leq \Delta_{\max}^i$ . By Claim 1,

$$\begin{aligned} (25) &= \sum_{j=0}^{j_{\max}(M)} \text{Reg}(\{\Delta_{S_t} \geq M, S_t \in \mathcal{S}_{i,j}, N_{i,j,t-1} < \ell_T(\Delta_{S_t}, 2^{-j})\}) \\ &\leq \sum_{j=0}^{j_{\max}(M)} \left( \ell_T(M, 2^{-j})M + \int_M^{\max\{\Delta_{\max}^i, M\}} \ell_T(x, 2^{-j}) dx \right) \\ &= \sum_{j=0}^{j_{\max}(M)} \left( \ell_T(M, 2^{-j})M + \int_M^{\Delta_{\max}^i} \ell_T(x, 2^{-j}) dx \right) \\ &= \sum_{j=0}^{j_{\max}(M)} \ell_T(M, 2^{-j})M + \int_M^{\Delta_{\max}^i} \sum_{j=0}^{j_{\max}(M)} \ell_T(x, 2^{-j}) dx. \end{aligned} \quad (26)$$

We then expand the notation  $\ell_T(\Delta, q)$  (c.f. Definition 8) with

$$\ell_T(\Delta, q) \leq \begin{cases} \frac{6 \ln T}{f^{-1}(\Delta)^2} + 1, & \text{if } q = 1, \\ \frac{72q \ln T}{f^{-1}(\Delta)^2} + 1, & \text{otherwise.} \end{cases}$$

So for any  $x \in [M, \Delta_{\max}^i]$ ,

$$\begin{aligned} \sum_{j=0}^{j_{\max}(M)} \ell_T(x, 2^{-j}) &= \ell_T(x, 1) + \sum_{j=1}^{j_{\max}(M)} \ell_T(x, 2^{-j}) \\ &\leq \left( \frac{6 \ln T}{f^{-1}(x)^2} + 1 \right) + \sum_{j=1}^{j_{\max}(M)} \left( \frac{72 \cdot 2^{-j} \ln T}{f^{-1}(x)^2} + 1 \right) \\ &= \frac{6 \ln T}{f^{-1}(x)^2} + \sum_{j=1}^{j_{\max}(M)} \frac{72 \cdot 2^{-j} \ln T}{f^{-1}(x)^2} + j_{\max}(M) + 1 \\ &\leq \frac{6 \ln T}{f^{-1}(x)^2} + \frac{72 \ln T}{f^{-1}(x)^2} + j_{\max}(M) + 1 \\ &= \frac{78 \ln T}{f^{-1}(x)^2} + j_{\max}(M) + 1. \end{aligned}$$

Then we continue (26) with

$$\begin{aligned} (26) &\leq \left( \frac{78 \ln T}{f^{-1}(M)^2} + j_{\max}(M) + 1 \right) \cdot M + \int_M^{\Delta_{\max}^i} \left( \frac{78 \ln T}{f^{-1}(x)^2} + j_{\max}(M) + 1 \right) dx \\ &= \frac{78 \ln T}{f^{-1}(M)^2} \cdot M + \int_M^{\Delta_{\max}^i} \frac{78 \ln T}{f^{-1}(x)^2} dx + (j_{\max}(M) + 1) \cdot \Delta_{\max}^i \\ &= 78 \ln T \left( \frac{M}{f^{-1}(M)^2} + \int_M^{\Delta_{\max}^i} \frac{1}{f^{-1}(x)^2} dx \right) + (j_{\max}(M) + 1) \cdot \Delta_{\max}^i. \end{aligned}$$

Hence the lemma holds.  $\square$

**Lemma 13.** For event-filtered regret

$$Reg(\{\Delta_{S_t} < M\}) + \sum_{i \in [m]} \sum_{j \geq 0} Reg(\{\Delta_{S_t} \geq M, S_t \in \mathcal{S}_{i,j}, N_{i,j,t-1} < \ell_T(\Delta_{S_t}, 2^{-j})\}), \quad (27)$$

(1) take  $M = \Delta_{\min}$  when  $\Delta_{\min} > 0$ ,

$$(27) \leq \sum_{i \in [m]} 78 \ln T \left( \frac{\Delta_{\min}^i}{f^{-1}(\Delta_{\min}^i)^2} + \int_{\Delta_{\min}^i}^{\Delta_{\max}^i} \frac{1}{f^{-1}(x)^2} dx \right) + m \cdot (j_{\max}(\Delta_{\min}) + 1) \cdot \Delta_{\max};$$

(2) if  $f(x) = ax$ , then take  $M = a\sqrt{156m \ln T/T}$ ,

$$(27) < 25a\sqrt{mT \ln T} + m \cdot (j_{\max}(a\sqrt{156m \ln T/T}) + 1) \cdot \Delta_{\max}.$$

*Proof.* (1) If  $\Delta_{S_t} < M = \Delta_{\min}$ , then  $\Delta_{S_t} = 0$ . So  $Reg(\{\Delta_{S_t} < M\}) \leq 0$ . For every  $i \in [m]$  and every integer  $j$ , we may replace  $M$  with  $\Delta_{\min}^i$  as below.

$$\begin{aligned} &Reg(\{\Delta_{S_t} \geq M, S_t \in \mathcal{S}_{i,j}, N_{i,j,t-1} < \ell_T(\Delta_{S_t}, 2^{-j})\}) \\ &= Reg(\{\Delta_{S_t} \geq \Delta_{\min}^i, S_t \in \mathcal{S}_{i,j}, N_{i,j,t-1} < \ell_T(\Delta_{S_t}, 2^{-j})\}) \\ &= Reg(\{\Delta_{S_t} \geq \Delta_{\min}^i, S_t \in \mathcal{S}_{i,j}, N_{i,j,t-1} < \ell_T(\Delta_{S_t}, 2^{-j})\}). \end{aligned} \quad (28)$$

Then apply Lemma 12 with  $M = \Delta_{\min}^i$ , we have

$$\begin{aligned}
 (27) &= \sum_{i \in [m]} \sum_{j \geq 0} \text{Reg}(\{\Delta_{S_t} \geq M, S_t \in \mathcal{S}_{i,j}, N_{i,j,t-1} < \ell_T(\Delta_{S_t}, 2^{-j})\}) \\
 &\leq \sum_{i \in [m]} \left[ 78 \ln T \left( \frac{\Delta_{\min}^i}{f^{-1}(\Delta_{\min}^i)^2} + \int_{\Delta_{\min}^i}^{\Delta_{\max}^i} \frac{1}{f^{-1}(x)^2} dx \right) + (j_{\max}(\Delta_{\min}^i) + 1) \cdot \Delta_{\max}^i \right] \\
 &\leq \sum_{i \in [m]} 78 \ln T \left( \frac{\Delta_{\min}^i}{f^{-1}(\Delta_{\min}^i)^2} + \int_{\Delta_{\min}^i}^{\Delta_{\max}^i} \frac{1}{f^{-1}(x)^2} dx \right) + m \cdot (j_{\max}(\Delta_{\min}) + 1) \cdot \Delta_{\max}.
 \end{aligned}$$

(2) By Lemma 12, for every arm  $i$ ,

$$\begin{aligned}
 &\sum_{j \geq 0} \text{Reg}(\{\Delta_{S_t} \geq M, S_t \in \mathcal{S}_{i,j}, N_{i,j,t-1} < \ell_T(\Delta_{S_t}, 2^{-j})\}) \\
 &\leq 78 \ln T \left( \frac{M}{f^{-1}(M)^2} + \int_M^{\Delta_{\max}^i} \frac{1}{f^{-1}(x)^2} dx \right) + (j_{\max}(M) + 1) \cdot \Delta_{\max}^i \\
 &= 78 \ln T \left( \frac{M}{(a^{-1}M)^2} + \int_M^{\Delta_{\max}^i} \frac{1}{(a^{-1}x)^2} dx \right) + (j_{\max}(M) + 1) \cdot \Delta_{\max}^i \\
 &= 78 \ln T \left( \frac{1}{a^{-2}M} + \int_M^{\Delta_{\max}^i} \frac{1}{a^{-2}x^2} dx \right) + (j_{\max}(M) + 1) \cdot \Delta_{\max}^i \\
 &\leq 78 \ln T \left( \frac{1}{a^{-2}M} + \frac{1}{a^{-2}M} \right) + (j_{\max}(M) + 1) \cdot \Delta_{\max}^i \\
 &= \frac{156 \ln T}{a^{-2}M} + (j_{\max}(M) + 1) \cdot \Delta_{\max}^i. \tag{29}
 \end{aligned}$$

$\text{Reg}(\{\Delta_{S_t} < M\}) < TM$  as the regret in each round is less than  $M$ . So by (29) and take  $M = a\sqrt{156m \ln T/T}$ ,

$$\begin{aligned}
 (27) &< TM + \frac{156m \ln T}{a^{-2}M} + m \cdot (j_{\max}(M) + 1) \cdot \Delta_{\max} \\
 &= a\sqrt{156mT \ln T} + a\sqrt{156mT \ln T} + m \cdot (j_{\max}(M) + 1) \cdot \Delta_{\max} \\
 &< 25a\sqrt{mT \ln T} + m \cdot (j_{\max}(a\sqrt{156m \ln T/T}) + 1) \cdot \Delta_{\max}. \quad \square
 \end{aligned}$$

*Proof of Theorem 6.* (1) Since  $\Delta_{\min} > 0$ , we can take  $M = \Delta_{\min}$ . By Lemma 13(1) and Claim 2, we continue Inequality (24) as below.

$$\begin{aligned}
 (24) &\leq \sum_{i \in [m]} 78 \ln T \left( \frac{\Delta_{\min}^i}{f^{-1}(\Delta_{\min}^i)^2} + \int_{\Delta_{\min}^i}^{\Delta_{\max}^i} \frac{1}{f^{-1}(x)^2} dx \right) + m \cdot (j_{\max}(\Delta_{\min}) + 1) \cdot \Delta_{\max} \\
 &\quad + m \cdot (j_{\max}(\Delta_{\min}) + 2) \cdot \frac{\pi^2}{6} \cdot \Delta_{\max} \\
 &= \sum_{i \in [m]} 78 \ln T \left( \frac{\Delta_{\min}^i}{f^{-1}(\Delta_{\min}^i)^2} + \int_{\Delta_{\min}^i}^{\Delta_{\max}^i} \frac{1}{f^{-1}(x)^2} dx \right) \\
 &\quad + m \cdot \left[ \left( \frac{\pi^2}{6} + 1 \right) \lceil -\log_2 f^{-1}(\Delta_{\min}) \rceil_0 + \frac{\pi^2}{3} + 1 \right] \cdot \Delta_{\max}.
 \end{aligned}$$

(2) Take  $M = a\sqrt{156m \ln T/T}$ , by Lemma 13(2) and Claim 2, we continue Inequality (24) as below.

$$\begin{aligned}
 (24) &\leq 25a\sqrt{mT \ln T} + m \cdot (j_{\max}(a\sqrt{156m \ln T/T}) + 1) \cdot \Delta_{\max} + m \cdot (j_{\max}(a\sqrt{156m \ln T/T}) + 2) \cdot \frac{\pi^2}{6} \cdot \Delta_{\max} \\
 &= 25a\sqrt{mT \ln T} + m \cdot \left[ \left( \frac{\pi^2}{6} + 1 \right) \lceil -\log_2(\sqrt{156m \ln T/T}) \rceil_0 + \frac{\pi^2}{3} + 1 \right] \cdot \Delta_{\max}.
 \end{aligned}$$

For (3) and (4) of Theorem 6, we bound the event-filtered regret as below, using Claims 3 and Lemma 6.

$$\begin{aligned}
 & \text{Reg}(\{\Delta_{S_t} \geq M, \forall i, N_{i,q_i^{S_t},t-1} \geq \ell_T(\Delta_{S_t}, q_i^{S_t})\}) \\
 & \leq \text{Reg}(\{\exists i, \rho_{i,t} \geq \frac{1}{4m}\}) + \text{Reg}(\{\Delta_{S_t} \geq M, \forall i, \rho_{i,t} \leq \frac{1}{4m}, N_{i,q_i^{S_t}} \geq \ell_T(\Delta_{S_t}, q_i^{S_t})\}). \\
 & \leq (24m^3 \ln T + m) \cdot \Delta_{\max} + m \cdot (j_{\max}(M) + 2) \cdot \frac{\pi^2}{6} \cdot \Delta_{\max}. \tag{30}
 \end{aligned}$$

(3) Since  $\Delta_{\min} > 0$ , we can take  $M = \Delta_{\min}$ . By Inequality (30) and Lemma 13(1), we continue Inequality (24) as below.

$$\begin{aligned}
 (24) & \leq \sum_{i \in [m]} 78 \ln T \left( \frac{\Delta_{\min}^i}{f^{-1}(\Delta_{\min}^i)^2} + \int_{\Delta_{\min}^i}^{\Delta_{\max}^i} \frac{1}{f^{-1}(x)^2} dx \right) + m \cdot (j_{\max}(\Delta_{\min}) + 1) \cdot \Delta_{\max} \\
 & \quad + (24m^3 \ln T + m) \cdot \Delta_{\max} + m \cdot (j_{\max}(\Delta_{\min}) + 2) \cdot \frac{\pi^2}{6} \cdot \Delta_{\max} \\
 & = \sum_{i \in [m]} 78 \ln T \left( \frac{\Delta_{\min}^i}{f^{-1}(\Delta_{\min}^i)^2} + \int_{\Delta_{\min}^i}^{\Delta_{\max}^i} \frac{1}{f^{-1}(x)^2} dx \right) \\
 & \quad + m \cdot \left[ 24m^2 \ln T + \left( \frac{\pi^2}{6} + 1 \right) \lceil -\log_2 f^{-1}(\Delta_{\min}) \rceil_0 + \frac{\pi^2}{3} + 2 \right] \cdot \Delta_{\max}.
 \end{aligned}$$

(4) Take  $M = a\sqrt{156m \ln T/T}$ , by Inequality (30) and Lemma 13(1), we continue Inequality (24) as below.

$$\begin{aligned}
 (24) & \leq 25a\sqrt{mT \ln T} + m \cdot (j_{\max}(a\sqrt{156m \ln T/T}) + 1) \cdot \Delta_{\max} \\
 & \quad + (24m^3 \ln T + m) \cdot \Delta_{\max} + m \cdot (j_{\max}(a\sqrt{156m \ln T/T}) + 2) \cdot \frac{\pi^2}{6} \cdot \Delta_{\max} \\
 & = 25a\sqrt{mT \ln T} + m \cdot \left[ 24m^2 \ln T + \left( \frac{\pi^2}{6} + 1 \right) \lceil -\log_2(\sqrt{156m \ln T/T}) \rceil_0 + \frac{\pi^2}{3} + 2 \right] \cdot \Delta_{\max}. \quad \square
 \end{aligned}$$

### D.2.2. PROOF DETAILS

In this subsection, we finish the remaining part of the proof, i.e. the proofs of the claims. We first prove the bound of sufficiently sampled part, namely Claims 2 & 3. To do so, we define two kinds of niceness, that the difference between  $\mu_i$  and  $\hat{\mu}_i$  is small enough and that  $T_i$  is large enough comparing with  $N_{i,j}$ , and then show that both kinds of niceness are satisfied with high probability and if so, it is impossible to play a bad action. We then prove Claim 1. In this subsection we assume  $M$  is already defined as a positive real number as in the proof of Theorem 6. Notations  $\hat{\mu}_t, \hat{\mu}_{i,t}, \bar{\mu}_t, \bar{\mu}_{i,t}$  denote the values of  $\hat{\mu}, \hat{\mu}_i, \bar{\mu}, \bar{\mu}_i$  at the end of round  $t$ , respectively.

We now prove the claims.

*Proof of Claim 2.* Explicitly,

$$\begin{aligned}
 \text{Reg}(\{\Delta_{S_t} \geq M, \forall i, N_{i,q_i^{S_t},t-1} \geq \ell_T(\Delta_{S_t}, q_i^{S_t})\}) & = \sum_{t=1}^T \mathbb{E}[\Delta_{S_t} \cdot \mathbb{I}\{\Delta_{S_t} \geq M, \forall i, N_{i,q_i^{S_t},t-1} \geq \ell_T(\Delta_{S_t}, q_i^{S_t})\}] \\
 & \leq \sum_{t=1}^T \Pr\{\Delta_{S_t} \geq M, \forall i, N_{i,q_i^{S_t},t-1} \geq \ell_T(\Delta_{S_t}, q_i^{S_t})\} \cdot \Delta_{\max}. \tag{31}
 \end{aligned}$$

We only need to bound  $\Pr\{\Delta_{S_t} \geq M, \forall i, N_{i,q_i^{S_t},t-1} \geq \ell_T(\Delta_{S_t}, q_i^{S_t})\}$ , i.e. the probability that for every  $i$ , there is  $N_{i,q_i^{S_t},t-1} \geq \ell_T(\Delta_{S_t}, q_i^{S_t})$ , but an action  $S_t$  with  $\Delta_{S_t} \geq M$  is still played. Let event  $\mathcal{E}_t = \{\Delta_{S_t} \geq M, \forall i, N_{i,q_i^{S_t},t-1} \geq \ell_T(\Delta_{S_t}, q_i^{S_t})\}$ . We now prove the claim that event  $\mathcal{E}_t$  is not empty only when  $\neg(\mathcal{N}_t^s \wedge \mathcal{N}_t^t)$ , or equivalently if both the sampling and triggering are nice at the beginning of round  $t$ , then event  $\mathcal{E}_t$  is empty. If the sampling is nice at the beginning of round  $t$ , then

$$\bar{\mu}_{i,t-1} = \min\{\hat{\mu}_{i,t-1} + \rho_{i,t}, 1\} \geq \mu_i.$$



By monotonicity,  $r_S(\bar{\mu}_{t-1}) \geq r_S(\mu)$  for every action  $S$ , so  $\text{opt}_{\bar{\mu}_{t-1}} \geq \text{opt}_\mu$ . As action  $S_t$  is chosen by Oracle with input  $\bar{\mu}_{t-1}$ , it must be that  $r_{S_t}(\bar{\mu}_{t-1}) = \text{opt}_{\bar{\mu}_{t-1}} \geq \text{opt}_\mu$ , so  $r_{S_t}(\bar{\mu}_{t-1}) - r_{S_t}(\mu) \geq \text{opt}_\mu - r_{S_t}(\mu) = \Delta_{S_t}$ . We are going to show the claim by assuming  $\mathcal{N}_t^s \wedge \mathcal{N}_t^l$  and showing  $\forall i, p_i^{S_t} |\bar{\mu}_{i,t-1} - \mu_i| < f^{-1}(\Delta_{S_t})$ , then by  $\infty$ -norm TPM bounded smoothness (Condition 8),  $r_{S_t}(\bar{\mu}_{t-1}) - r_{S_t}(\mu) < \Delta_{S_t}$ , which is a contradiction. Note that here we do need strict inequality “ $<$ ” instead of “ $\leq$ ” when applying Condition 8. This can be done because  $i$  has at most  $m$  choices and the bounded smoothness function  $f$  is continuous and strictly increasing, so we can use a small enough  $\varepsilon > 0$  such that  $\forall i, p_i^{S_t} |\bar{\mu}_{i,t-1} - \mu_i| \leq f^{-1}(\Delta_{S_t} - \varepsilon)$ , and thus  $r_{S_t}(\bar{\mu}_{t-1}) - r_{S_t}(\mu) \leq \Delta_{S_t} - \varepsilon < \Delta_{S_t}$ .

Below we omit  $S_t$  from  $\Delta_{S_t}$ ,  $p_i^{S_t}$  and  $q_i^{S_t}$ . If  $f^{-1}(\Delta) > p_i$ , then  $p_i |\bar{\mu}_{i,t-1} - \mu_i| \leq p_i |1 - 0| < f^{-1}(\Delta)$  without any dependency on sampling. If  $f^{-1}(\Delta) \leq p_i$ , then  $q_i \leq 2^{\lceil -\log_2 f^{-1}(\Delta) \rceil} \leq 2^{j_{\max}(M)}$ . When the sampling is nice (Definition 4),  $\bar{\mu}_{i,t-1} \leq \hat{\mu}_{i,t-1} + \rho_{i,t} \leq \mu_i + 2\rho_{i,t}$ . On the other hand,  $|\bar{\mu}_{i,t-1} - \mu_i| \leq |1 - 0| = 1$ . When the triggering is nice (Definition 7), if  $\sqrt{\frac{6 \ln t}{\frac{1}{3} N_{i,q_i,t-1} \cdot q_i}} \leq 1$ , then  $2\rho_{i,t} \leq \sqrt{\frac{6 \ln t}{\frac{1}{3} N_{i,q_i,t-1} \cdot q_i}}$ . So regardless whether  $\sqrt{\frac{6 \ln t}{\frac{1}{3} N_{i,q_i,t-1} \cdot q_i}} \leq 1$ ,  $|\bar{\mu}_{i,t-1} - \mu_i| \leq \sqrt{\frac{6 \ln t}{\frac{1}{3} N_{i,q_i,t-1} \cdot q_i}}$ . Event  $\mathcal{E}_t$  implies that  $N_{i,q_i,t-1} \geq \ell_T(\Delta, q_i) \geq \ell_t(\Delta, q_i)$  (since  $t \leq T$ ). So

$$p_i |\bar{\mu}_{i,t-1} - \mu_i| \leq p_i \sqrt{\frac{6 \ln t}{\frac{1}{3} N_{i,q_i,t-1} \cdot q_i}} \leq p_i \sqrt{\frac{6 \ln t}{\frac{1}{3} \ell_t(\Delta, q_i) \cdot q_i}} < p_i \sqrt{\frac{6 \ln t}{\frac{1}{3} \frac{72 q_i \ln t}{f^{-1}(\Delta)^2} \cdot q_i}} = p_i \sqrt{\frac{f^{-1}(\Delta)^2}{4 q_i^2}} \leq p_i \sqrt{\frac{f^{-1}(\Delta)^2}{p_i^2}} = f^{-1}(\Delta).$$

Hence, the claim holds.

The claim implies that  $\Pr\{\mathcal{E}_t\} \leq \Pr\{\neg(\mathcal{N}_t^s \wedge \mathcal{N}_t^l)\} \leq \Pr\{\neg\mathcal{N}_t^s\} + \Pr\{\neg\mathcal{N}_t^l\}$ . By Lemmas 4 and 5, we have  $\Pr\{\mathcal{E}\} \leq (2 + j_{\max}(M))mt^{-2}$ . Plugging it into Inequality (31), we have

$$\begin{aligned} \text{Reg}(\{\Delta_{S_t} \geq M, \forall i, N_{i,q_i^{S_t},t-1} \geq \ell_T(\Delta_{S_t}, q_i^{S_t})\}) &\leq \sum_{t=1}^T (2 + j_{\max}(M))mt^{-2} \cdot \Delta_{\max} \\ &\leq m \cdot (\lceil -\log_2 f^{-1}(M) \rceil_0 + 2) \cdot \frac{\pi^2}{6} \cdot \Delta_{\max}. \quad \square \end{aligned}$$

*Proof of Claim 3.* The proof of Claim 3 is very similar with the proof of Claim 2. First, we have

$$\begin{aligned} &\text{Reg}(\{\Delta_{S_t} \geq M, \forall i, \rho_{i,t} \leq \frac{1}{4m}, N_{i,q_i^{S_t},t-1} \geq \ell_T(\Delta_{S_t}, q_i^{S_t})\}) \\ &= \sum_{t=1}^T \mathbb{E}[\Delta_{S_t} \cdot \mathbb{I}\{\Delta_{S_t} \geq M, \forall i, \rho_{i,t} \leq \frac{1}{4m}, N_{i,q_i^{S_t},t-1} \geq \ell_T(\Delta_{S_t}, q_i^{S_t})\}] \\ &\leq \sum_{t=1}^T \Pr\{\Delta_{S_t} \geq M, \forall i, \rho_{i,t} \leq \frac{1}{4m}, N_{i,q_i^{S_t},t-1} \geq \ell_T(\Delta_{S_t}, q_i^{S_t})\} \cdot \Delta_{\max}. \end{aligned}$$

Then, be the proof of Claim 1, if  $\mathcal{N}_t^s \wedge \mathcal{N}_t^l$ , then  $|\bar{\mu}_i - \mu_i| < p_i^{-1} f^{-1}(\Delta)$  for every arm  $i$ . With the additional condition  $\rho_{i,t} \leq \frac{1}{4m}$ , we can apply RTPM bounded smoothness (Condition 9), and thus  $r_{S_t}(\bar{\mu}) - r_{S_t}(\mu) < \Delta_{S_t}$  and the claim holds.  $\square$

*Proof of Claim 4.* Let  $x$  be any real number that  $x \geq M > 0$ . In any round when an action  $S$  with  $S \in \mathcal{S}_{i,j}$  is played,  $N_{i,j}$  is increased by 1. So

$$\sum_{t=1}^T \Pr\{S_t \in \mathcal{S}_{i,j}, N_{i,j,t-1} < \ell_T(x, 2^{-j})\} \leq \ell_T(x, 2^{-j}).$$

If we add an additional restriction  $\Delta_{S_t} \geq x$ , the probability will not increase, so

$$\sum_{t=1}^T \Pr\{\Delta_{S_t} \geq x, S_t \in \mathcal{S}_{i,j}, N_{i,j,t-1} < \ell_T(x, 2^{-j})\} \leq \ell_T(x, 2^{-j}).$$

We use the shorthand  $\mathcal{E}_{i,j}^{S_t}$  to denote the event  $\{S_t \in \mathcal{S}_{i,j}, N_{i,j,t-1} < \ell_T(x, 2^{-j})\}$ . Suppose  $X$  is a non-negative random variable with  $\Pr\{X \geq M\} = p$  and  $\Pr\{X = 0\} = 1 - p$ . Then by the basic principal on expectation, we have

$$\mathbb{E}[X] = \int_0^{+\infty} \Pr\{X \geq x\} dx = \int_0^M \Pr\{X \geq x\} dx + \int_M^{+\infty} \Pr\{X \geq x\} dx = pM + \int_M^{+\infty} \Pr\{X \geq x\} dx.$$

Applying the above, we have

$$\begin{aligned}
 \text{Reg}(\{\Delta_{S_t} \geq M\} \cap \mathcal{E}_{i,j}^{S_t}) &= \sum_{t=1}^T \mathbb{E}[\mathbb{I}(\{\Delta_{S_t} \geq M\} \cap \mathcal{E}_{i,j}^{S_t}) \cdot \Delta_{S_t}] \\
 &= \sum_{t=1}^T \left( \Pr[\{\Delta_{S_t} \geq M\} \cap \mathcal{E}_{i,j}^{S_t}] \cdot M + \int_M^{+\infty} \Pr[\{\Delta_{S_t} \geq x\} \cap \mathcal{E}_{i,j}^{S_t}] dx \right) \\
 &= \sum_{t=1}^T \Pr[\{\Delta_{S_t} \geq M\} \cap \mathcal{E}_{i,j}^{S_t}] \cdot M + \int_M^{+\infty} \sum_{t=1}^T \Pr[\{\Delta_{S_t} \geq x\} \cap \mathcal{E}_{i,j}^{S_t}] dx \\
 &= \sum_{t=1}^T \Pr[\{\Delta_{S_t} \geq M\} \cap \mathcal{E}_{i,j}^{S_t}] \cdot M + \int_M^{\max\{\Delta_{\max}^i, M\}} \sum_{t=1}^T \Pr[\{\Delta_{S_t} \geq x\} \cap \mathcal{E}_{i,j}^{S_t}] dx \\
 &\leq \ell_T(M, 2^{-j})M + \int_M^{\max\{\Delta_{\max}^i, M\}} \ell_T(x, 2^{-j}) dx. \quad \square
 \end{aligned}$$

### D.3. Comparison between 1-norm and $\infty$ -norm

In this paper, we give upper bounds of regret for CMAB-T problems that satisfy TPM/RTPM bounded smoothness with 1-norm or with  $\infty$ -norm. We emphasize Theorem 1&2 and Theorem 6 do not imply each other. For clarity, we use  $a_1$  and  $a_\infty$  in place of  $a$  in bounded smoothness function  $f(x) = ax$ . If a CMAB-T problem instance satisfies TPM/RTPM bounded smoothness with 1-norm with  $f(x) = a_1x$ , then it also satisfies TPM/RTPM bounded smoothness with  $\infty$ -norm with  $f(x) = a_\infty x$ , where  $a_\infty = Ka_1$ . Conversely, if a CMAB-T problem instance satisfies TPM/RTPM bounded smoothness with  $\infty$ -norm with  $f(x) = a_\infty x$ , then it also satisfies TPM/RTPM bounded smoothness with 1-norm with  $f(x) = a_1x$ , where  $a_1 = a_\infty$ . For distribution-dependent upper bound, according to Theorems 1, 2 and 6, we have  $O(\frac{a_\infty^2 m \ln T}{\Delta})$  and  $O(\frac{a_1^2 Km \ln T}{\Delta})$ . For a problem instance that satisfies TPM/RTPM bounded smoothness with 1-norm with  $f(x) = a_1x$ , if we use the bound for  $\infty$ -norm, the result will be  $O(\frac{a_1^2 K^2 m \ln T}{\Delta})$ . For a problem instance that satisfies TPM/RTPM bounded smoothness with  $\infty$ -norm with  $f(x) = a_\infty x$ , if we use the bound for 1-norm, the result will be  $O(\frac{a_\infty^2 Km \ln T}{\Delta})$ . Both give an additional  $K$  factor. It is similar for distribution-independent bound, which will have an additional  $\sqrt{K}$  factor in both cases.

## E. Omitted Proofs in Section 5 (Lower Bound Proofs)

### E.1. Proof of Theorem 3

We prove the theorem by reducing classical MAB to this CMAB-T game instance by Algorithm 2. For convenience, we define Bernoulli random variable  $\gamma_t = \mathbb{I}\{\tau_t(S_{i_t}, X^{(t)}) = \{i_t\}\}$ , where  $S_{i_t}$  is the action played in round  $t$ , and thus  $\gamma_t$  is an indicator representing whether a base arm is triggered in round  $t$ . Moreover, to distinguish the environment outcome in MAB and CMAB-T in the reduction, we use  $\tilde{X}^{(t_{\text{MAB}})}$  to denote the environment outcome in round  $t_{\text{MAB}}$  of MAB, and  $X^{(t)}$  to denote the environment outcome in round  $t$  of CMAB-T.

Figure 1 shows the structure of reduction. Algorithm 2 adapts the CMAB-T algorithm to an MAB algorithm. Conversely, it also adapts the MAB instance to the corresponding CMAB-T instance. Thus when Algorithm 2 runs, we have one MAB instance and one CMAB-T instance running simultaneously. Let  $T_{\text{CMAB}}$  be the total number of rounds in the CMAB-T instance and  $T_{\text{MAB}}$  be the total number of rounds in the MAB instance. For convenience, we use  $t$  to refer to the index of rounds in CMAB-T, while  $t_{\text{MAB}}$  is the index of rounds in MAB. In Algorithm 2, we fix  $T_{\text{CMAB}}$  and thus  $T_{\text{MAB}}$  is a random variable. We have  $T_{\text{MAB}} = \sum_{t=1}^{T_{\text{CMAB}}} \gamma_t$ . So  $\mathbb{E}[T_{\text{MAB}}] = pT_{\text{CMAB}}$  and we have following lemma about the distribution of  $T_{\text{MAB}}$ .

**Lemma 14.** *If  $pT_{\text{CMAB}} \geq 6$ , then  $\Pr[T_{\text{MAB}} \geq \frac{1}{2}pT_{\text{CMAB}}] \geq \frac{1}{2}$ .*

**Algorithm 2** Reduce MAB to CMAB-T

**Input:**  $m, T_{\text{CMAB}}, p$   $\{m$  is the number of arms,  $T_{\text{CMAB}}$  is the number of rounds in CMAB, and  $p$  is triggering probability. $\}$ 

```

1: for  $t = 1, \dots, T_{\text{CMAB}}$  do
2:   sample  $\gamma_t$  i.i.d. from Bernoulli distribution  $B_p$ 
3: end for
4:  $\mathcal{H} \leftarrow \emptyset; t_{\text{MAB}} \leftarrow 0$ 
5: for  $t = 1, \dots, T_{\text{CMAB}}$  do
6:    $S_{i_t} \leftarrow \text{CMAB-Oracle}(\mathcal{H})$   $\{\text{Oracle decides the CMAB-T action based on the execution history}\}$ 
7:   if  $\gamma_t = 1$  then
8:      $t_{\text{MAB}} \leftarrow t_{\text{MAB}} + 1$ 
9:     In MAB, play arm  $i_t$  in round  $t_{\text{MAB}}$ , obtain feedback  $\tilde{X}_{i_t}^{(t_{\text{MAB}})}$ 
10:    In CMAB-T,  $i_t$  is triggered with feedback  $X_{i_t}^{(t)} = \tilde{X}_{i_t}^{(t_{\text{MAB}})}$ , and set reward as  $p^{-1} X_{i_t}^{(t)}$ 
11:     $\mathcal{H} \leftarrow \text{Append}(\mathcal{H}, (S_{i_t}, \{i_t\}, X_{i_t}^{(t)}))$   $\{\{i_t\}$  is the set of triggered arms $\}$ 
12:   else
13:      $\{\gamma_t = 0$ , and MAB is not played in this case $\}$ 
14:     In CMAB-T, no arm is triggered, and the reward is 0
15:      $\mathcal{H} \leftarrow \text{Append}(\mathcal{H}, (S_{i_t}, \emptyset, -))$   $\{\text{triggering set is empty, so no feedback}\}$ 
16:   end if
17: end for  $\{\text{In the end, } T_{\text{MAB}} = t_{\text{MAB}}\}$ 
    
```

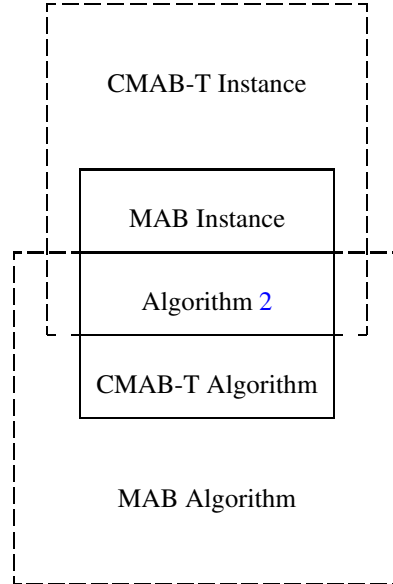


Figure 1. Reduction Structure

*Proof.*  $T_{\text{MAB}} = \sum_{t=1}^{T_{\text{CMAB}}} \gamma_t$ . By multiplicative Chernoff bound (Fact 2),

$$\Pr[T_{\text{MAB}} \geq \frac{1}{2}pT_{\text{CMAB}}] \geq 1 - \left( \frac{e^{-\frac{1}{2}}}{(\frac{1}{2})^{\frac{1}{2}}} \right)^{pT_{\text{CMAB}}} \geq \frac{1}{2},$$

when  $pT_{\text{CMAB}} \geq 6$ .

$$\Pr[T_{\text{MAB}} \geq \frac{1}{2}pT_{\text{CMAB}}] \geq 1 - \left( e^{-\frac{1}{8}pT_{\text{CMAB}}} \right) \geq \frac{1}{2},$$

when  $pT_{\text{CMAB}} \geq 6$ .

In the following, we overload the notation  $\mathcal{D}$  to also represent a probabilistic distribution of the environment instance (a.k.a. outcome distribution)  $D$ , and use  $D \sim \mathcal{D}$  to represent a random environment instance  $D$  drawn from the distribution  $\mathcal{D}$ .

**Lemma 15.** *Consider a random MAB environment instance  $D$  drawn from a distribution  $\mathcal{D}$ . Assume we have a lower bound  $L(T_{\text{MAB}})$  of expected regret, i.e. for every natural number  $T_{\text{MAB}}$ , any MAB algorithm  $A$  has expected regret*

$$\mathbb{E}_{D \sim \mathcal{D}} [\text{Reg}_{\text{MAB}, D}^A(T_{\text{MAB}})] \geq L(T_{\text{MAB}}).$$

*Then consider the corresponding CMAB-T environment instance  $D$ . For every natural number  $T_{\text{CMAB}} \geq 5p^{-1}$ , any CMAB-T algorithm  $A$  has expected regret*

$$\mathbb{E}_{D \sim \mathcal{D}} [\text{Reg}_{\text{CMAB}, D}^A(T_{\text{CMAB}})] \geq \frac{1}{2}p^{-1}L\left(\frac{1}{2}pT_{\text{CMAB}}\right). \quad (32)$$

*Proof.* Without loss of generality, we may assume  $L(T)$  is non-decreasing, as regret of any strategy increases as  $T$  increases.

We prove the lemma using the reduction described above. We run Algorithm 2 with  $A$  be the CMAB-T oracle and  $D$  be the environment instance. Let  $\gamma$  be the vector  $(\gamma_1, \gamma_2, \dots, \gamma_{T_{\text{CMAB}}})$ . Every possible value of  $\gamma$  parameterizes Algorithm 2 into an algorithm plays MAB problem for  $T_{\text{MAB}} = \sum_{t=1}^{T_{\text{CMAB}}} \gamma_t$  rounds. We denote this MAB algorithm with  $A_\gamma$ . By our assumption,  $\mathbb{E}_{D \sim \mathcal{D}} [\text{Reg}_{\text{MAB}, D}^{A_\gamma}(T_{\text{MAB}})] \geq L(T_{\text{MAB}})$ .

Then we compare the regret in both cases. For a given distribution  $D$ , let  $\mu_{i,D} = \mathbb{E}_{X \sim D}[X_i]$  and  $\mu_D^* = \max_i \mu_{i,D}$ . For MAB problem and every  $\gamma$ ,

$$\begin{aligned} \mathbb{E}_{D \sim \mathcal{D}} [\text{Reg}_{\text{MAB}, D}^{A_\gamma}(T_{\text{MAB}})] &= \mathbb{E}_{D \sim \mathcal{D}} \left[ T_{\text{MAB}} \cdot \mu_D^* - \mathbb{E} \left[ \sum_{t=1}^{T_{\text{CMAB}}} \gamma_t X_{i_t} \right] \right] \\ &= \mathbb{E}_{D \sim \mathcal{D}} \left[ \mathbb{E} \left[ \sum_{t=1}^{T_{\text{CMAB}}} \gamma_t (\mu_D^* - X_{i_t}) \right] \right] \\ &= \mathbb{E}_{D \sim \mathcal{D}} \left[ \mathbb{E} \left[ \sum_{t=1}^{T_{\text{CMAB}}} \gamma_t (\mu_D^* - \mu_{i_t, D}) \right] \right], \end{aligned}$$

where the inner expectation is taken over the rest randomness, including the randomness of  $i_t$ , which is based on the

random feedback history and the possible randomness of algorithm  $A_\gamma$ . For CMAB-T, we have

$$\begin{aligned}
 \mathbb{E}_{D \sim \mathcal{D}} [Reg_{\text{CMAB}, D}^A(T_{\text{CMAB}})] &= \mathbb{E}_{D \sim \mathcal{D}} \left[ T_{\text{CMAB}} \cdot \mu_D^* - \mathbb{E}_{\gamma \sim B_p^{T_{\text{CMAB}}}} \left[ \mathbb{E} \left[ \sum_{t=1}^{T_{\text{CMAB}}} \gamma_t p^{-1} X_{i_t} \right] \right] \right] \\
 &= \mathbb{E}_{D \sim \mathcal{D}} \left[ T_{\text{CMAB}} \cdot \mu_D^* - \mathbb{E}_{\gamma \sim B_p^{T_{\text{CMAB}}}} \left[ \mathbb{E} \left[ \sum_{t=1}^{T_{\text{CMAB}}} \gamma_t p^{-1} \mu_{i_t, D} \right] \right] \right] \\
 &= \mathbb{E}_{D \sim \mathcal{D}} \left[ p T_{\text{CMAB}} \cdot p^{-1} \mu_D^* - \mathbb{E}_{\gamma \sim B_p^{T_{\text{CMAB}}}} \left[ \mathbb{E} \left[ \sum_{t=1}^{T_{\text{CMAB}}} \gamma_t p^{-1} \mu_{i_t, D} \right] \right] \right] \\
 &= \mathbb{E}_{D \sim \mathcal{D}} \left[ \mathbb{E}_{\gamma \sim B_p^{T_{\text{CMAB}}}} \left[ \sum_{t=1}^{T_{\text{CMAB}}} \gamma_t p^{-1} \mu_D^* \right] - \mathbb{E}_{\gamma \sim B_p^{T_{\text{CMAB}}}} \left[ \mathbb{E} \left[ \sum_{t=1}^{T_{\text{CMAB}}} \gamma_t p^{-1} \mu_{i_t, D} \right] \right] \right] \\
 &= p^{-1} \mathbb{E}_{D \sim \mathcal{D}, \gamma \sim B_p^{T_{\text{CMAB}}}} \left[ \mathbb{E} \left[ \sum_{t=1}^{T_{\text{CMAB}}} \gamma_t (\mu^* - \mu_{i_t, D}) \right] \right],
 \end{aligned}$$

where the innermost expectation is taken over the rest randomness such as the randomness of  $i_t$ . Therefore

$$\mathbb{E}_{D \sim \mathcal{D}} [Reg_{\text{CMAB}, D}^A(T_{\text{CMAB}})] = p^{-1} \mathbb{E}_{D \sim \mathcal{D}, \gamma \sim B_p^{T_{\text{CMAB}}}} [Reg_{\text{MAB}, D}^{A_\gamma}(T_{\text{MAB}})].$$

Calculation above also shows  $\mathbb{E}_{D \sim \mathcal{D}} [Reg_{\text{MAB}, D}^{A_\gamma}(T_{\text{MAB}})] \geq 0$ . And by monotonicity of  $L(T)$ ,

$$\begin{aligned}
 \mathbb{E}_D [Reg_{\text{CMAB}, D}^A(T_{\text{CMAB}})] &= p^{-1} \mathbb{E}_{D, \gamma} [Reg_{\text{MAB}, D}^{A_\gamma}(T_{\text{MAB}})] \\
 &\geq p^{-1} \mathbb{E}_{D, \gamma} [\mathbb{I}\{T_{\text{MAB}} \geq \frac{1}{2} p T_{\text{CMAB}}\} Reg_{\text{MAB}, D}^{A_\gamma}(T_{\text{MAB}})] \\
 &\geq p^{-1} \mathbb{E}_{D, \gamma} [\mathbb{I}\{T_{\text{MAB}} \geq \frac{1}{2} p T_{\text{CMAB}}\} L(\frac{1}{2} p T_{\text{CMAB}})] \\
 &= p^{-1} \Pr_{D, \gamma} \{T_{\text{MAB}} \geq \frac{1}{2} p T_{\text{CMAB}}\} L(\frac{1}{2} p T_{\text{CMAB}}) \\
 &\geq \frac{1}{2} p^{-1} L(\frac{1}{2} p T_{\text{CMAB}}). \quad \square
 \end{aligned}$$

**Lemma 16.** Let  $m$  be the number of arms and  $T$  be the number of rounds. Let  $\varepsilon = \frac{1}{10} \sqrt{m/T}$ . Then define the family of MAB outcome distributions  $\mathcal{D} = \{D_1, \dots, D_m\}$  with

$$\Pr_{D_j} \{X_i = 1\} = \begin{cases} \frac{1}{2} & \text{if } i \neq j \\ \frac{1}{2} + \varepsilon & \text{if } i = j \end{cases}.$$

Let  $D$  be a random environment instance uniformly drawn from  $\mathcal{D}$ , then for any MAB algorithm  $A$ ,

$$\mathbb{E}_{D \sim \mathcal{D}} [Reg_{\text{MAB}, D}^A(T)] \geq \frac{\varepsilon T}{6} = \frac{1}{60} \sqrt{mT}.$$

*Proof of Theorem 3.* Let  $\mathcal{D}$  be the family of outcome distributions defined in Lemma 16, and  $D$  is uniformly drawn from  $\mathcal{D}$ . Applying the result of Lemma 16 to Lemma 15, with  $L(T) = \frac{1}{60} \sqrt{mT}$  in Lemma 15, we have

$$\begin{aligned}
 \mathbb{E}_{D \sim \mathcal{D}} [Reg_{\text{CMAB}, D}^A(T)] &\geq \frac{1}{2} p^{-1} L(\frac{1}{2} p T) \\
 &= \frac{1}{2} p^{-1} \cdot \frac{1}{60} \sqrt{\frac{1}{2} m p T} \\
 &> \frac{1}{170} \sqrt{\frac{mT}{p}}.
 \end{aligned}$$

Since  $D$  is uniformly drawn from  $\mathcal{D}$ , then there must exists a  $D \in \mathcal{D}$  such that

$$\text{Reg}_{\text{CMAB}, D}^A(T) \geq \frac{1}{170} \sqrt{\frac{mT}{p}}. \quad \square$$

It is easy to show corresponding CMAB-T problem satisfies original bounded smoothness (Condition 7) with  $f(x) = x$ . So the theorem above gives an example that the upper bound in (Chen et al., 2016b) is tight up to a  $O(\sqrt{\log T})$  factor.

Note that in our reduced CMAB-T instance, we have  $r_{S_i}(\mu) = p \cdot p^{-1}\mu_i = \mu_i$ . Therefore, if we use the original bounded smoothness condition (Condition 7), we would set bounded smoothness function  $f(x) = x$ . In this case, the minimum positive triggering probability  $p^* = p$ , and thus from the results in (Chen et al., 2016b) would lead to regret upper bound  $O\left(\sqrt{\frac{mT \log T}{p}}\right)$ . This means the regret upper bound of CUCB in (Chen et al., 2016b) is tight up to a  $O(\sqrt{\log T})$  factor. Alternatively, if we want to use the TPM bounded smoothness condition (Condition 8), then if  $p|\mu_i - \mu'_i| \leq \Lambda$  for all  $i \in [m]$ , we have  $|r_{S_i}(\mu) - r_{S_i}(\mu')| = |\mu_i - \mu'_i| \leq p^{-1}\Lambda$ . This means we need to use  $f(x) = p^{-1}x$  as the bounded smoothness function. Then by Theorem 6(2), we have upper bound  $O(p^{-1}\sqrt{mT \log T})$ , not as good as the upper bound we obtain from Condition 7. Therefore, Theorem 3 shows that the regret upper bound in (Chen et al., 2016b) is tight, when the new TPM bounded smoothness condition cannot be effectively applied.

## E.2. Proof of Theorem 4

*Proof of Theorem 4.* We regard this kind of CMAB-T problem instances as a variant of classical MAB, that each arm gives three possible outcomes, 0, 1, and  $\perp$ . Denote these arms with random variables  $X'_1, \dots, X'_n$ . The reward is  $p^{-1}$  times of the outcome if the outcome is 0 or 1, while the reward is 0 if the outcome is  $\perp$ . This variant is equivalent to the CMAB-T instances: Outcome  $X'_i = \perp$  corresponds to Bernoulli base arm  $X_i$  in CMAB-T not being triggered, outcome  $X'_i = 1$  or 0 corresponds to Bernoulli base arm  $X_i$  being triggered and  $X_i = 1$  or 0, respectively. Thus  $\Pr[X'_i = \perp] = 1 - p$ ,  $\Pr[X'_i = 0] = p(1 - \mu_i)$ , and  $\Pr[X'_i = 1] = p\mu_i$ , where  $p$  is the triggering probability and  $\mu_i$  is the expectation of  $X_i$ .

Let  $X$  and  $Y$  be random variables whose values are in the same finite set  $V$ . Define the KL-divergence

$$\text{kl}(X, Y) = \sum_{x \in V} \Pr\{X = x\} \ln \frac{\Pr\{X = x\}}{\Pr\{Y = x\}}.$$

For example the KL-divergence between  $X'_1$  and  $X'_2$  is

$$\begin{aligned} \text{kl}(X'_1, X'_2) &= \Pr\{X'_1 = \perp\} \ln \frac{\Pr\{X'_1 = \perp\}}{\Pr\{X'_2 = \perp\}} + \Pr\{X'_1 = 0\} \ln \frac{\Pr\{X'_1 = 0\}}{\Pr\{X'_2 = 0\}} + \Pr\{X'_1 = 1\} \ln \frac{\Pr\{X'_1 = 1\}}{\Pr\{X'_2 = 1\}} \\ &= (1 - p) \ln \frac{1 - p}{1 - p} + p(1 - \mu_1) \ln \frac{p(1 - \mu_1)}{p(1 - \mu_2)} + p\mu_1 \ln \frac{p\mu_1}{p\mu_2} \\ &= 0 + p(1 - \mu_1) \ln \frac{1 - \mu_1}{1 - \mu_2} + p\mu_1 \ln \frac{\mu_1}{\mu_2} \\ &= p \cdot \left[ (1 - \mu_1) \ln \frac{1 - \mu_1}{1 - \mu_2} + \mu_1 \ln \frac{\mu_1}{\mu_2} \right] \\ &= p \cdot \text{kl}(X_1, X_2). \end{aligned} \quad \square$$

Thus, intuitively it takes  $p^{-1}$  times more rounds to differentiate  $X'_1$  and  $X'_2$  than  $X_1$  and  $X_2$ , which is stated formally in theorem below.

*Proof.* The analysis is generalized from the case that the arms are Bernoulli random variables. For an arm  $i$ , we use  $N_i(T)$  to denote the number of times the arm  $i$  is played in  $T$  rounds. For each non-optimal arm  $i$ , i.e.  $\mu_i < \mu^* < 1$ , we show

$$\liminf_{T \rightarrow +\infty} \frac{\mathbb{E}[N_i(T)]}{\ln T} \geq \frac{p^{-1}}{\text{kl}(X_i, X_{i^*})} = \frac{1}{\text{kl}(X'_i, X'_{i^*})}. \quad (33)$$

Then by formula

$$\text{Reg}_{\mu}^A(T) = \sum_{i: \mu_i < \mu^*} \mathbb{E}[N_i(T)] \Delta_i,$$



the theorem holds.

Without loss of generality, we may assume arm 1 is an optimal arm and arm 2 is non-optimal. We prove Eq. (33) for arm 2 and then the inequality holds for every arm. Consider that if we replace arm 2 with a fictional arm  $2'$ , which has an expectation  $\mu_{2'}$  slightly greater than  $\mu_1$ , then arm 1 will become non-optimal and strategy  $A$  will play arm 1 for  $o(n^a)$  times for any  $a > 0$ . So strategy  $A$  must play arm 2 for enough times, to differentiate from arm  $2'$ .

Formally, let  $\varepsilon > 0$  be any positive real number. Let  $\mu_{2'}$  be a real number such that  $\mu_{2'} > \mu_1$  and

$$\text{kl}(X_2, X_{2'}) = (1 - \mu_2) \ln \frac{1 - \mu_2}{1 - \mu_{2'}} + \mu_2 \ln \frac{\mu_2}{\mu_{2'}} < (1 + \varepsilon) \text{kl}(X_2, X_1). \quad (34)$$

There exists such  $\mu_{2'}$ , because the left hand side of (34) is continuous as a function of  $\mu_{2'}$ . We use  $\mathbb{E}'$  and  $\Pr'$  to denote expectation and probability in the circumstance that arm  $X_2$  is replaced by arm  $X_{2'}$ .

We define the empirical KL-divergence after the first  $s$  samples of the arm  $2/2'$ ,

$$\widehat{\text{kl}}_s = \sum_{t=1}^s Y_t,$$

where

$$Y_t = \begin{cases} \ln \frac{1 - \mu_2}{1 - \mu_{2'}}, & \text{if } X'_{2,t} = 0, \\ \ln \frac{\mu_2}{\mu_{2'}}, & \text{if } X'_{2,t} = 1, \\ 0, & \text{if } X'_{2,t} = \perp. \end{cases}$$

and  $X'_{2,t}$  is result of the  $t$ -th sample of arm  $2/2'$ . Note that  $(Y_t)$  are independent and  $\mathbb{E}[Y_t] = \text{kl}(X'_2, X_{2'})$ .

First we prove

$$\Pr \left\{ N_2(T) < \frac{1 - \varepsilon}{\text{kl}(X'_2, X_{2'})} \ln T \wedge \widehat{\text{kl}}_{N_2(T)} \leq \left(1 - \frac{\varepsilon}{2}\right) \ln T \right\} = o(1). \quad (35)$$

We use the shorthands

$$C_T = \left\{ N_2(T) < \frac{1 - \varepsilon}{\text{kl}(X'_2, X_{2'})} \ln T \wedge \widehat{\text{kl}}_{N_2(T)} \leq \left(1 - \frac{\varepsilon}{2}\right) \ln T \right\}, \quad (36)$$

and

$$f_T = \frac{1 - \varepsilon}{\text{kl}(X'_2, X_{2'})} \ln T.$$

If arm 2 is replaced by arm  $2'$ , we have

$$\Pr' \{C_T\} \leq \Pr' \{N_2(T) < f_T\} \leq \frac{\mathbb{E}'[T - N_2(T)]}{T - f_T},$$

where the second inequality is due to Markov's inequality. Recall the definition of consistent strategy, as  $2'$  is the only optimal arm, we have  $\mathbb{E}'[T - N_2(T)] = o(T^{\frac{\varepsilon}{2}})$ . And by  $T - f_T = \Omega(T)$ ,  $\Pr' \{C_T\} = o(T^{\frac{\varepsilon}{2}-1})$ . Then we use the property of KL-divergence

$$\Pr \{C_T\} = \mathbb{E}' \left[ \mathbb{I}\{C_T\} \cdot \exp \left( \widehat{\text{kl}}_{N_2(T)} \right) \right],$$

then

$$\Pr \{C_T\} = \mathbb{E}' \left[ \mathbb{I}\{C_T\} \cdot \exp \left( \widehat{\text{kl}}_{N_2(T)} \right) \right] \leq \Pr' \{C_T\} \cdot \exp \left[ \left(1 - \frac{\varepsilon}{2}\right) \ln T \right] = \Pr' \{C_T\} \cdot T^{1-\frac{\varepsilon}{2}} = o(1).$$

Second, we prove

$$\Pr \left\{ N_2(T) < f_T \wedge \widehat{\text{kl}}_{T_2(T)} > \left(1 - \frac{\varepsilon}{2}\right) \ln T \right\} = o(1). \quad (37)$$

We have

$$\begin{aligned} \Pr \left\{ N_2(T) < f_T \wedge \widehat{\text{kl}}_{N_2(T)} > \left(1 - \frac{\varepsilon}{2}\right) \ln T \right\} &\leq \Pr \left\{ N_2(T) < f_T \wedge \max_{s \leq f_T} \widehat{\text{kl}}_s > \left(1 - \frac{\varepsilon}{2}\right) \ln T \right\} \\ &\leq \Pr \left\{ \max_{s \leq f_T} \widehat{\text{kl}}_s > \left(1 - \frac{\varepsilon}{2}\right) \ln T \right\}. \end{aligned}$$

Recall the definition of  $\widehat{\text{kl}}_s$ , which is a summation of independent random variables with the same distribution over a finite support, whose expectation is  $\text{kl}(X'_2, X'_{2'})$ . So we apply the maximal version of the strong law of large numbers, and then (37) holds, as  $f_T \cdot \text{kl}(X'_2, X'_{2'}) = (1 - \varepsilon) \ln T$ .

In conclusion, combining Eq. (35) and (37), we have  $\Pr\{N_2(T) < f_T\} = o(1)$ , implying

$$\begin{aligned} \mathbb{E}[N_2(T)] &\geq (1 - o(1)) \cdot f_T \\ &= (1 - o(1)) \cdot \frac{1 - \varepsilon}{\text{kl}(X'_2, X'_{2'})} \ln T \\ &\geq (1 - o(1)) \cdot \frac{1 - \varepsilon}{1 + \varepsilon} \frac{\ln T}{\text{kl}(X'_2, X'_1)}. \end{aligned}$$

Then (33) holds, as  $\varepsilon$  can be any positive real number, and thus the theorem holds.  $\square$

## F. Proof in Section 4.1

### F.1. 1-Norm TPM Bounded Smoothness Condition for the Influence Maximization Bandit on Forests

Let  $\tilde{C}$  be the number of nodes in the largest component of the bidirectional forest in the influence maximization bandit instance.

**Lemma 17.** *Influence maximization bandit on a bidirectional forest satisfies 1-norm TPM bounded smoothness with bounded smoothness constant  $B = \tilde{C}$ .*

*Proof.* For each node  $v$ , let  $r_S^v(\mu)$  be the expected reward if we only give  $v$  with a weight of 1 and other nodes have weight 0. In other words,  $r_S^v(\mu)$  is the probability that  $v$  is activated. Then  $r_S(\mu) = \sum_{v \in V} r_S^v(\mu)$ . Let  $\text{Comp}(v)$  be the component that contains node  $v$ .

We then show  $|r_S^v(\mu) - r_S^v(\mu')| \leq \sum_{i \in \text{Comp}(v)} p_i^{\mu, S} |\mu_i - \mu'_i|$ . Consider the inward directed tree rooted at  $v$ , which is a directional subgraph of the original one. If the graph is just this directed tree, the expected rewards  $r_S^v(\mu)$  and  $r_S^v(\mu')$  are unchanged while the probability of triggering each edge is not larger than the original. So we only need to prove the bounded smoothness for this directional tree. Just like Lemma 1, we change the expectations of edges from the root to leaves, then the probability of triggering an edge is unchanged before we change the expectation of that edge. Then when changing the expectation of each edge from  $\mu_i$  to  $\mu'_i$ ,  $r_S^v(\mu)$  changes by at most  $p_i^{\mu, S} |\mu_i - \mu'_i|$ . So  $|r_S^v(\mu) - r_S^v(\mu')| \leq \sum_{i \in \text{Comp}(v)} p_i^{\mu, S} |\mu_i - \mu'_i|$ .

Finally, we add it together for every node as

$$|r_S(\mu) - r_S(\mu')| \leq \sum_{v \in V} |r_S^v(\mu) - r_S^v(\mu')| \leq \sum_{v \in V} \sum_{i \in \text{Comp}(v)} p_i^{\mu, S} |\mu_i - \mu'_i|.$$

Each edge  $i$  is contained in a component of at most  $\tilde{C}$  nodes. So  $|r_S(\mu) - r_S(\mu')| \leq \tilde{C} \sum_{i \in [m]} p_i^{\mu, S} |\mu_i - \mu'_i|$ .  $\square$